Multilevel Distance Labelings for Paths and Cycles

Daphne Der-Fen Liu *
Department of Mathematics
California State University, Los Angeles
Los Angeles, CA 90032, USA
Email: dliu@calstatela.edu

Xuding Zhu †
Department of Applied Mathematics
National Sun Yat-sen University
Kaohsiung, Taiwan 80424
Email: zhu@math.nsysu.edu.tw

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Abstract

For a graph G, let $\mathrm{diam}(G)$ denote the diameter of G. For any two vertices u and v in G, let $d(u, v)$ denote the distance between u and v. A multilevel distance labeling (or distance labeling) for G is a function f that assigns to each vertex of G a non-negative integer such that for any vertices u and v, $|f(u) - f(v)| \geq \mathrm{diam}(G) - d_G(u, v) + 1$. The span of f is the largest number in $f(V)$. The radio number of G, denoted by $rn(G)$, is the minimum span of a distance labeling for G. In this paper, we completely determine the radio numbers for paths and cycles.

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1 Introduction

multilevel distance labeling can be regarded as an extension of distance two labeling which is motivated by the channel assignment problem introduced by Hale [10]. For a set of given cities (or stations), the task is to assign to each city a channel, which is a non-negative integer, so that interference is prohibited, and the span of the channels assigned is minimized.

Usually, the level of interference between any two stations is closely related to the geographic locations of the stations – the closer are the stations the stronger is the interference. Suppose we consider two levels of interference, major and minor. Major interference occurs between two *very close* stations; to avoid it, the channels assigned to a pair of very close stations have to be at least two apart. Minor interference occurs between *close* stations; to avoid it, the channels assigned to close stations have to be different.

To model this problem, we construct a graph G by representing each station by a vertex and connecting two vertices by an edge if the geographical locations of the corresponding stations are *very close*. Two close stations are represented by, in the corresponding graph G, a pair of vertices that are distance two apart.

Let $d_G(u,v)$ denote the distance (the shortest length of a path) between u and v in G (or simply $d(u,v)$ when G is clear in the context). Thus, for a graph G, a distance two labeling (or $L(2,1)$-labeling) with span k is a function, $f : V(G) \to \{0,1,2,\cdots,k\}$, such that the following are satisfied: 1) $|f(x)-f(y)| \geq 2$ if $d(x,y)=1$; and 2) $|f(x)-f(y)| \geq 1$ if $d(x,y)=2$.

Distance two labeling has been studied extensively in the past decade (cf. [1, 2, 5, 6, 7, 8, 9, 11, 12, 13, 14, 15]). One of the main research focuses has been the λ-number for a graph G, denoted by $\lambda(G)$, which is the smallest span k of a distance two labeling for G.

Practically, interference among channels might go beyond two levels. We consider interference levels from one through the largest possible value – the *diameter* of G, denoted by $\mathrm{diam}(G)$, which is the largest distance between two vertices of G.

A *multilevel distance labeling* (or *distance labeling* for short), with span k, is a function $f : V(G) \to \{0, 1, 2, \cdots, k\}$, so that for any vertices u and v,

$$|f(u) - f(v)| \geq \mathrm{diam}(G) - d_G(u, v) + 1.$$

The *radio number* (as suggested by the FM radio frequency assignment [4]) for G, denoted by $rn(G)$, is the minimum span of a distance labeling for G. Note that if $\mathrm{diam}(G) = 2$, then distance two labeling coincides with multilevel distance labeling, and in this case, $\lambda(G) = rn(G)$.

Besides its motivation from the channel assignment problem, distance labeling itself is an interesting, relatively new notion in graph coloring, and worthy of further investigation for its own sake. It is surprising that determining the radio number seems a difficult problem even for some basic families of graphs. For instance, the radio number for paths and cycles has been studied by Chartrand et al. [3] and Chartrand, Erwin, and Zhang [4]. In [4] and [3], some bounds of the radio numbers for paths and cycles, respectively, were presented, while the exact values remained unknown at that time.

In this article, we completely determine the radio numbers for paths and cycles. Note that, to be consistent with distance two labelings, we allow 0 to be used as a color (or channel). However, in [4, 3], only positive integers can be used as colors. Therefore, the radio number defined in this article is *one less* than the radio number defined in [4, 3]. Being consistent, throughout the article, we make necessary adjustments, reflecting this "one less" difference, for all the results quoted from [4, 3].

2 The Radio Number for Paths

Let P_n be the path on n vertices. Chartrand, Erwin and Zhang [4] proved the following upper bounds for $rn(P_n)$:

Theorem 1 *[4] For any positive integer n,*

$$rn(P_n) \le \begin{cases} 2k^2 + k, & \text{if } n = 2k+1; \\ 2(k^2 - k) + 1, & \text{if } n = 2k. \end{cases}$$

Moreover, the bound is sharp when $n \le 5$.

In this section, we completely settle the radio numbers for paths. We first prove the following Lemma.

Lemma 2 *Let P_n be a path with vertex set $V(P_n) = \{v_1, v_2, \cdots, v_n\}$, in which $v_i \sim v_{i+1}$ for $i = 1, 2, \cdots, n-1$. Let f be an assignment of distinct non-negative integers to $V(P_n)$. Let $(x_1, x_2, \cdots, x_n)$ be the ordering of $V(P_n)$ such that $f(x_i) < f(x_{i+1})$. The following three statements are equivalent.*

(1) For any $1 \le i \le n-2$, $\min\{d(x_i, x_{i+1}), d(x_{i+1}, x_{i+2})\} \le n/2$.

(2) If $f(x_{i+1}) - f(x_i) \ge n - d(x_i, x_{i+1})$ for all $1 \le i \le n-1$, then f is a distance labeling.

(3) If $f(x_{i+1}) - f(x_i) = n - d(x_i, x_{i+1})$ for all $1 \le i \le n-1$, then f is a distance labeling.

Proof. Note that $\text{diam}(P_n) = n - 1$.
(1) ⇒ (2) Assume (1) For any $1 \le i \le n-2$, $\min\{d(x_i, x_{i+1}), d(x_{i+1}, x_{i+2})\} \le n/2$, and (2) $f(x_{i+1}) - f(x_i) \ge n - d(x_i, x_{i+1})$ for all $1 \le i \le n-1$. We need to show that for any $i \ne j$, $|f(x_i) - f(x_j)| \ge n - d(x_i, x_j)$.

For each $i = 1, 2, \cdots, n-1$, set

$$f_i = f(x_{i+1}) - f(x_i).$$

Assume $i < j$. Then

$$f(x_j) - f(x_i) = f_i + f_{i+1} + \cdots + f_{j-1}.$$

Assumptions (1) and (2) imply that $f_i \geq n - d(x_i, x_{i+1})$, $f_{i+1} \geq n - d(x_{i+1}, x_{i+2})$, and for any i,

$$\max\{f_i, f_{i+1}\} \geq n/2.$$

Thus, if $j \geq i+4$, then $f(x_j) - f(x_i) \geq n > n - d(x_i, x_j)$, and we are done. It suffices to consider the cases that $j = i+2$ or $j = i+3$.

Assume $j = i+2$. Without loss of generality, we may assume that $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$, and hence $d(x_{i+1}, x_{i+2}) \leq n/2$. Since $d(x_i, x_{i+2}) \geq d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2})$, we have

$$\begin{aligned} f(x_j) - f(x_i) &= f_i + f_{i+1} \\ &\geq (n - d(x_i, x_{i+1})) + (n - d(x_{i+1}, x_{i+2})) \\ &= 2n - 2d(x_{i+1}, x_{i+2}) - (d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2})) \\ &\geq n - d(x_i, x_{i+2}). \end{aligned}$$

Assume $j = i+3$. If the sum of some pair of the distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, and $d(x_{i+2}, x_{i+3})$ is at most n, then $f(x_{i+3}) - f(x_i) = f_i + f_{i+1} + f_{i+2} \geq n$, so we are done.

Thus, we assume that the sum of every pair of the distances $d(x_i, x_{i+1})$, $d(x_{i+1}, x_{i+2})$, and $d(x_{i+2}, x_{i+3})$ is greater than n. This implies that

$$d(x_{i+1}, x_{i+2}) \leq n/2 \text{ and } d(x_i, x_{i+1}), d(x_{i+2}, x_{i+3}) > n/2.$$

Let $x_i = v_a$, $x_{i+1} = v_b$, $x_{i+2} = v_c$, $x_{i+3} = v_d$. Let m and m' be, respectively, the maximum and the minimum of $\{a, b, c, d\}$. Then $\{m, m'\} = \{a, d\}$. For otherwise, say $m' = b$, then we have $b < c < d$, implying that $d(x_{i+1}, x_{i+2}) + d(x_{i+2}, x_{i+3}) \leq n$, in contrary to our assumption. Hence, one has

$$d(x_i, x_{i+3}) = d(x_i, x_{i+1}) + d(x_{i+2}, x_{i+3}) - d(x_{i+1}, x_{i+2}) > n/2.$$

So, $f(x_{i+3}) - f(x_i) = f_i + f_{i+1} + f_{i+2} > f_{i+1} \geq n/2 > n - d(x_i, x_{i+3})$.

(2) ⇒ (3) Trivial.

(3) ⇒ (1) Let $f(x_1) = 0$, and let $f(x_i) = f(x_{i-1}) + n - d(x_i, x_{i+1})$ for all i. By (3), f is a distance labeling of P_n. Assume, to the contrary of (1), that there is an index i such that

$$\min\{d(x_i, x_{i+1}), d(x_{i+1}, x_{i+2})\} > n/2.$$

Without loss of generality, we assume that $d(x_i, x_{i+1}) \geq d(x_{i+1}, x_{i+2})$. Then

$$d(x_i, x_{i+2}) = d(x_i, x_{i+1}) - d(x_{i+1}, x_{i+2}),$$

and thus

$$\begin{aligned} f(x_{i+2}) - f(x_i) &= n - d(x_i, x_{i+1}) + n - d(x_{i+1}, x_{i+2}) \\ &= 2n - 2(d(x_{i+1}, x_{i+2})) - d(x_i, x_{i+2}) \\ &< n - d(x_i, x_{i+2}), \end{aligned}$$

contrary to the assumption that f is a distance labeling. ∎

Theorem 3 *For any* $n \geq 4$,

$$rn(P_n) = \begin{cases} 2k^2 + 2, & \text{if } n = 2k + 1; \\ 2k(k-1) + 1, & \text{if } n = 2k. \end{cases}$$

Proof. Note that, for even paths, by Theorem 1, it suffices to show that $rn(P_{2k}) \geq 2k(k-1)+1$. However, for completeness, we present a proof here without using Theorem 1.

First, we show that $rn(P_{2k+1}) \leq 2k^2+2$ and $rn(P_{2k}) \leq 2k(k-1)+1$. Assume $P_{2k+1} = (v_1, v_2, \cdots, v_{2k+1})$, where $v_i \sim v_{i+1}$. Order the vertices of P_{2k+1} as follows:

$$v_k, v_{k+k}, v_1, v_{1+k}, v_{1+k+k}, v_3, v_{3+k}, v_4, v_{4+k}, v_5, v_{5+k}, \cdots, v_{k-1}, v_{k-1+k}, v_2, v_{2+k}.$$

Rename the vertices of P in the above ordering by $x_1, x_2, \cdots, x_{2k+1}$. Namely, let $x_1 = v_k$, $x_2 = v_{k+k}$, $\cdots$, $x_{2k+1} = v_{2+k}$.

Let f be the mapping defined as $f(x_1) = 0$, and for $i = 2, 3, \cdots, 2k+1$,

$$f(x_i) = f(x_{i-1}) + 2k + 1 - d(x_{i-1}, x_i).$$

It is easy to verify that the ordering and the mapping f satisfy the conditions of Lemma 2 (1) and (3). Therefore f is a distance labeling of P_{2k+1}.

It remains to show that $f(x_{2k+1}) = 2k^2 + 2$. By definition,

$$\begin{aligned} f(x_{2k+1}) &= \sum_{i=1}^{2k} [2k+1-d(x_i, x_{i+1})] \\ &= 2k(2k+1) - \sum_{i=1}^{2k} d(x_i, x_{i+1}). \end{aligned}$$

Thus, it suffices to show that

$$\sum_{i=1}^{2k} d(x_i, x_{i+1}) = 2k^2 + 2k - 2.$$

Note that if $x_i = v_j$ and $x_{i+1} = v_{j'}$, then $d(x_i, x_{i+1}) = |j - j'|$, which is equal to either $j - j'$ or $j' - j$, whichever is positive. By replacing each term $d(x_i, x_{i+1})$ with the corresponding $j - j'$ or $j' - j$, whichever is positive, we obtain a summation whose entries are $\pm j$ for $j \in \{1, 2, \cdots, 2k+1\}$.

For the ordering above, if $j \leq k$, then the vertex preceding v_j is $v_{j'}$ for some $j' \geq k+2$, and the vertex following v_j is $v_{j''}$ for some $j'' \geq k+1$. Therefore, for each $1 \leq j \leq k$, whenever $\pm j$ occurs in the summation above, it occurs as a $-j$. Similarly, if $k+2 \leq j \leq 2k+1$, then whenever $\pm j$ occurs in the summation it occurs as a $+j$. The number $k+1$ occurs once as $+(k+1)$ and once as $-(k+1)$. Also it is easy to see that each j occurs twice in the summation, except that each of $j = k$ and $j = k+2$ occurs only once in the summation. Hence, we have

$$\begin{aligned} \sum_{i=1}^{2k} d(x_i, x_{i+1}) &= 2(\sum_{j=k+2}^{2k+1} j - \sum_{j=1}^{k} j) - (k+2-k) \\ &= 2k^2 + 2k - 2. \end{aligned}$$

The case for even paths is similar. Order the vertices of P_{2k} as follows:

$$v_k, v_{k+k}, v_2, v_{2+k}, v_3, v_{3+k}, \cdots, v_{k-1}, v_{k-1+k}, v_1, v_{1+k}.$$

Rename the vertices so that the ordering above is $x_1, x_2, \cdots, x_{2k}$. Namely, let $x_1 = v_k, x_2 = v_{k+k}, \cdots, x_{2k} = v_{1+k}$.

Let f be the mapping defined as $f(x_1) = 0$, and for $i = 2, 3, \cdots, 2k$,

$$f(x_i) = f(x_{i-1}) + 2k - d(x_{i-1}, x_i).$$

Then the ordering and the mapping f satisfy the conditions of Lemma 2 (1) and (3). Therefore f is a distance labeling of P_{2k}.

Similarly, in the summation $\sum_{i=1}^{2k-1} d(x_i, x_{i+1})$, each $j \in \{1, 2, \cdots, k-1\}$ occurs twice as $-j$, k occurs once as a $-k$, each of $j \in \{k+2, k+3, \cdots, 2k\}$ occurs twice as $+j$, and $k+1$ occurs once as a $+(k+1)$. Therefore,

$$\begin{aligned} \sum_{i=1}^{2k-1} d(x_i, x_{i+1}) &= 2(\sum_{j=k+2}^{2k} j - \sum_{j=1}^{k-1} j) + k + 1 - k \\ &= 2k^2 - 1. \end{aligned}$$

This implies,

$$\begin{aligned} f(x_{2k}) &= \sum_{i=1}^{2k-1} [2k - d(x_i, x_{i+1})] \\ &= 2k(2k-1) - \sum_{i=1}^{2k-1} d(x_i, x_{i+1}) \\ &= 4k^2 - 2k - 2k^2 + 1 \\ &= 2k(k-1) + 1. \end{aligned}$$

Next, we show that $rn(P_{2k+1}) \geq 2k^2 + 2$. Let f be a distance labeling of P_{2k+1}. Order the vertices of P_{2k+1} as $x_1, x_2, \cdots, x_{2k+1}$ such that $f(x_i) < f(x_{i+1})$ for all i. Assume $x_i = v_{\sigma(i)}$. Then σ is a permutation of $\{1, 2, \cdots, 2k+1\}$. We shall prove that $f(x_{2k+1}) \geq 2k^2 + 2$.

By definition, $f(x_1) \geq 0$ and $f(x_i) \geq f(x_{i-1}) + 2k + 1 - d(x_{i-1}, x_i)$ for $i = 2, 3, \cdots, 2k+1$. Thus

$$\begin{aligned} f(x_{2k+1}) &\geq \sum_{i=1}^{2k} [2k + 1 - d(x_i, x_{i+1})] \\ &= 2k(2k+1) - \sum_{i=1}^{2k} d(x_i, x_{i+1}). \end{aligned}$$

If $\sum_{i=1}^{2k} d(x_i, x_{i+1}) \leq 2k^2 + 2k - 2$, then $f(x_{2k+1}) \geq 2k^2 + 2$, and we are done. Hence, assume $\sum_{i=1}^{2k} d(x_i, x_{i+1}) > 2k^2 + 2k - 2$.

Claim. If $\sum_{i=1}^{2k} d(x_i, x_{i+1}) > 2k^2+2k-2$, then $\sum_{i=1}^{2k} d(x_i, x_{i+1}) = 2k^2+2k-1$ and there is an index i such that $f(x_{i+1}) - f(x_i) \geq n - d(x_{i+1}, x_i) + 1$.

Proof of Claim. Note that $d(x_i, x_{i+1})$ is equal to either $\sigma(i) - \sigma(i+1)$ or $\sigma(i+1) - \sigma(i)$, whichever is positive. By replacing each term $d(x_i, x_{i+1})$ with the corresponding $\sigma(i) - \sigma(i+1)$ or $\sigma(i+1) - \sigma(i)$, whichever is positive, we obtain a summation whose entries are $\pm j$ for $j \in \{1, 2, \cdots, 2k+1\}$.

All together, there are $4k$ terms in the summation $\sum_{i=1}^{2k} d(x_i, x_{i+1})$, half of them positive and half negative. Each $j \in \{1, 2, \cdots, 2k+1\}$ occurs as

$\pm j$ exactly twice in the summation, except for two values which each occurs only once.

To maximize the summation $\sum_{i=1}^{2k} d(x_i, x_{i+1})$, one needs to minimize the absolute values for the negative terms while maximize the values of the positive terms. It is easy to verify that there are two combinations achieving the maximum summation:

Case 1) Each of the numbers in $\{k+2, k+3, k+4, \cdots, 2k+1\}$ occurs twice as a positive, each of $\{1, 2, \cdots, k-1\}$ occurs twice as a negative, and each of k and $k+1$ occurs once as a negative.

Case 2) Each of the numbers in $\{k+3, k+4, \cdots, 2k+1\}$ occurs twice as a positive, each of $\{1, 2, \cdots, k\}$ occurs twice as a negative, and each of $k+1$ and $k+2$ occurs once as a positive.

In both cases, we have

$$\sum_{i=1}^{2k} d(x_i, x_{i+1}) = 2k^2 + 2k - 1.$$

In Case 1, we must have $\{\sigma(1), \sigma(2k+1)\} = \{k+1, k\}$. Moreover, $\sigma(i) \geq k+2$ if and only if $\sigma(i+1) \leq k+1$. In particular, if $\sigma(i) = 1$, then $\sigma(i-1) \geq k+2$ and $\sigma(i+1) \geq k+2$. This violates (1) in Lemma 2. As f is a distance labeling, it follows from Lemma 2 (3) that there exists some i such that $f(x_{i+1}) - f(x_i) \geq n - d(x_i, x_{i+1}) + 1$.

In Case 2, we must have $\{\sigma(1), \sigma(2k+1)\} = \{k+1, k+2\}$. Moreover, $\sigma(i) \geq k+1$ if and only if $\sigma(i+1) \leq k$. In particular, if $\sigma(i) = 2k+1$, then $\sigma(i-1) \leq k$ and $\sigma(i+1) \leq k$. Again, this violates (1) in Lemma 2, and it follows from Lemma 2 (3) that there exists some i such that $f(x_{i+1}) - f(x_i) \geq n - d(x_i, x_{i+1}) + 1$. □

By some calculation, it follows from Claim that if $\sum_{i=1}^{2k} d(x_i, x_{i+1}) > 2k^2 + 2k - 2$, we also have $f(x_{2k+1}) \geq 2k^2 + 2$, completing the proof for odd paths.

We now show that $rn(P_{2k}) \geq 2(k^2 - k) + 1$. Let f be a distance labeling of P_{2k}. Let $x_1, x_2, \cdots, x_{2k}$ be the ordering of the vertices of P_{2k} such that $f(x_i) < f(x_{i+1})$ for all i. Then

$$\begin{aligned} f(x_{2k}) &\geq \sum_{i=1}^{2k-1} [2k - d(x_i, x_{i+1})] \\ &= 2k(2k-1) - \sum_{i=1}^{2k-1} d(x_i, x_{i+1}). \end{aligned}$$

Similarly, in the summation $\sum_{i=1}^{2k-1} d(x_i, x_{i+1})$, each $j \in \{1, 2, \cdots, 2k\}$ occurs as twice as $\pm j$, except for two values which each occurs only once. Moreover, $2k-1$ of the terms are positive and $2k-1$ of them are negative. Thus to maximize the summation subject to the constraint, each number in $\{1, 2, \cdots, k-1\}$ occurs twice as negative terms, and each number in $\{k+2, k+3, \cdots, 2k\}$ occurs twice as positive terms, while k and $k+1$ occurs once, respectively, as a negative term and a positive term. Hence, we have

$$\sum_{i=1}^{2k-1} d(x_i, x_{i+1}) \leq 2(k^2 - 1) + 1,$$

implying

$$f(x_{2k}) \geq 2k(2k-1) - 2(k^2-1) - 1 \geq 2k(k-1) + 1.$$

∎

3 The Radio Number for Cycles

Let C_n denote the cycle on n vertices. Chartrand et al. [3] proved the following bounds for $rn(C_n)$:

Theorem 4 *[3] For $k \geq 3$,*

$$rn(C_n) \leq \begin{cases} k^2, & \text{if } n = 2k+1; \\ k^2 - k + 1, & \text{if } n = 2k. \end{cases}$$

Moreover, $rn(C_n) \geq 3\lceil \frac{n}{2} - 1 \rceil - 1$, for $n \geq 6$.

In this section, we completely determine the radio number for cycles. For any integer $n \geq 3$, let

$$\phi(n) = \begin{cases} k+1, & \text{if } n = 4k+1; \\ k+2, & \text{if } n = 4k+r \text{ for some } r = 0, 2, 3. \end{cases}$$

Theorem 5 *Let C_n be the n-vertex cycle, $n \geq 3$. Then*

$$rn(C_n) = \begin{cases} \frac{n-2}{2}\phi(n) + 1, & \text{if } n \equiv 0, 2 \pmod 4; \\ \frac{n-1}{2}\phi(n), & \text{if } n \equiv 1, 3 \pmod 4. \end{cases}$$

First we prove that the desired numbers in Theorem 5 are lower bounds for $rn(C_n)$. Assume $V(C_n) = \{v_0, v_1, v_2, \cdots, v_{n-1}\}$, where $v_i \sim v_{i+1}$ and $v_{n-1} \sim v_0$. Let f be a distance labeling for C_n. We order the vertices of $V(C_n)$ by $x_0, x_1, x_2, \cdots, x_{n-1}$ with $f(x_i) < f(x_{i+1})$.

Denote $d = \text{diam}(C_n)$. Then $d = \lfloor n/2 \rfloor$. For $i = 0, 1, 2, \cdots, n-2$, set

$$d_i = d(x_i, x_{i+1}) \text{ and } f_i = f(x_{i+1}) - f(x_i).$$

By definition, $f_i \geq d - d_i + 1$ for all i.

To proceed the proof of Theorem 5, we need the following two results.

Lemma 6 *For any $0 \leq i \leq n-3$, $f_i + f_{i+1} \geq \phi(n)$.*

Proof. Assume to the contrary that for some i, $f_i + f_{i+1} \leq \phi(n) - 1$. Then $f_i, f_{i+1} \leq \phi(n) - 2$. So, we have $d_i \geq d - f_i + 1 \geq d - \phi(n) + 3$ and $d_{i+1} \geq d - \phi(n) + 3$, implying that $d_i, d_{i+1} > d/2$. Therefore, $d(x_i, x_{i+2})$ is equal to either $|d_i - d_{i+1}|$ or $n - (d_i + d_{i+1})$. In the former case, $d(x_i, x_{i+2}) \leq d - (d - \phi(n) + 3) = \phi(n) - 3$, implying that

$$f_i + f_{i+1} = f(x_{i+2}) - f(x_i) \geq d - (\phi(n) - 3) + 1 \geq \phi(n),$$

contrary to our assumption.

If it is the latter case, then by definition all of the following hold:

$$\begin{aligned} f(x_{i+1}) - f(x_i) &\geq d - d_i + 1, \\ f(x_{i+2}) - f(x_{i+1}) &\geq d - d_{i+1} + 1, \\ f(x_{i+2}) - f(x_i) &\geq d - (n - d_i - d_{i+1}) + 1. \end{aligned}$$

Hence, $2(f(x_{i+2}) - f(x_i)) \geq 3d - n + 3$. Easy calculation shows that $f_i + f_{i+1} = f(x_{i+2}) - f(x_i) \geq \phi(n)$, a contradiction. ∎

Corollary 7 *For any integer $n \geq 3$,*

$$rn(C_n) \geq \begin{cases} \frac{n-2}{2}\phi(n) + 1, & \text{if } n \equiv 0, 2 \pmod 4; \\ \frac{n-1}{2}\phi(n), & \text{if } n \equiv 1, 3 \pmod 4. \end{cases}$$

Proof. If $n = 4k$ or $n = 4k + 2$, by Lemma 6, the span of a distance labeling f for C_n is:

$$f(x_{n-1}) = \sum_{i=0}^{n-2} f_i = \sum_{i=0}^{(n-4)/2} (f_{2i} + f_{2i+1}) + f_{n-2} \geq \frac{n-2}{2}\phi(n) + 1.$$

If $n = 4k + 1$ or $n = 4k + 3$, by Lemma 6, the span of a distance labeling f for C_n is:

$$f(x_{n-1}) = \sum_{i=0}^{n-2} f_i = \sum_{i=0}^{(n-3)/2} (f_{2i} + f_{2i+1}) \geq \frac{n-1}{2}\phi(n).$$

∎

To complete the proof of Theorem 5, it remains to find distance labelings for C_n with spans equal to the desired numbers. We consider four cases. For each case, we present a distance labeling f of C_n, achieving the bound.

In each of the four cases, the labeling is generated by two sequences, the *distance gap sequence*

$$D = (d_0, d_1, d_2, d_3, \cdots, d_{n-2})$$

and the *color gap sequence*

$$F = (f_0, f_1, f_2, \cdots f_{n-2}).$$

The distance gap sequence, in which each $d_i \leq d$ is a positive integer, is used to generate an ordering of the vertices of C_n. Let $\tau : \{0, 1, \cdots, n-1\} \to \{0, 1, \cdots, n-1\}$ be defined as $\tau(0) = 0$ and

$$\tau(i+1) = \tau(i) + d_i \pmod{n}.$$

We will show that for each of the distance sequences given below, the corresponding τ is a permutation. Let $x_i = v_{\tau(i)}$ for $i = 0, 1, 2 \cdots, n-1$. Then $x_0, x_1, \cdots, x_{n-1}$ is an ordering of the vertices of C_n. Since $1 \leq d_i \leq d$ for each i, we have $d(x_i, x_{i+1}) = d_i$.

The color gap sequence is used to assign labels to the vertices of C_n. Let f be the labeling defined by $f(x_0) = 0$, and for $i \geq 1$,

$$f(x_{i+1}) = f(x_i) + f_i.$$

Since $f_i = f(x_{i+1}) - f(x_i)$ and $d(x_i, x_{i+1}) = d_i$, to show that f is indeed a distance labeling, it suffices to prove that all the following hold, for any i:

1) τ is a permutation,

2) $f_i \geq d - d_i + 1$,

3) $f_i + f_{i+1} \geq d - d(x_i, x_{i+2}) + 1$,

4) $f_i + f_{i+1} + f_{i+2} \geq d - d(x_i, x_{i+3}) + 1$,

5) $f_i + f_{i+1} + f_{i+2} + f_{i+3} \geq d$.

For all the labelings given below, 5) is trivial, 2) is obvious, 3) and 4) are also easy to verify. In all the cases, we sketch a proof for 1), and leave it to the readers to verify 2) to 5).

Case 1. $n = 4k$ In this case, $d = 2k$. The distance gap sequence D is given by:

$$d_i = \begin{cases} 2k, & \text{if } i \text{ is even;} \\ k, & \text{if } i \equiv 1 \pmod 4; \\ k+1, & \text{if } i \equiv 3 \pmod 4. \end{cases}$$

The color gap sequence F is given by:

$$f_i = \begin{cases} 1, & \text{if } i \text{ is even;} \\ k+1, & \text{if } i \text{ is odd.} \end{cases}$$

Then we have, for $i = 0, 1, 2, \cdots, k-1$,

$$\begin{array}{lcll} \tau(4i) & = & 2ik + i & \pmod n, \\ \tau(4i+1) & = & (2i+2)k + i & \pmod n, \\ \tau(4i+2) & = & (2i+3)k + i & \pmod n, \\ \tau(4i+3) & = & (2i+1)k + i & \pmod n. \end{array}$$

We prove that τ is a permutation. Assume to the contrary that $\tau(4i+j) = \tau(4i'+j')$ for some $i, i' \in \{0, 1, 2, \cdots, k-1\}$ and $j, j' \in \{0, 1, 2, 3\}$ with $4i + j < 4i' + j'$. Then, clearly $i < i'$ and

$$(2i+t)k + i \equiv (2i'+t')k + i' \pmod n \text{ for some } t, t' = 0, 1, 2, 3.$$

Therefore, we have $2(i'-i)k + (t'-t)k \equiv i - i' \pmod n$, which is impossible, as $0 < i' - i < k$ and $2(i'-i)k + (t'-t)k \equiv sk \pmod n$ for some integer s.

The span of f is equal to $f_0 + f_1 + f_2 + \cdots + f_{n-2} = (k+2)(2k-1) + 1$.

$\boxed{\text{Case 2. } n = 4k+2}$ In this case, $d = 2k+1$. The distance gap sequence D is defined by:

$$d_i = \begin{cases} 2k+1, & \text{if } i \text{ is even;} \\ k+1, & \text{if } i \text{ is odd.} \end{cases}$$

The color gap sequence F is defined by

$$f_i = \begin{cases} 1, & \text{if } i \text{ is even;} \\ k+1, & \text{if } i \text{ is odd.} \end{cases}$$

Hence, for $i = 0, 1, \cdots, 2k$, we have

$$\begin{array}{lcll} \tau(2i) & = & i(3k+2) & \pmod{n}, \\ \tau(2i+1) & = & i(3k+2) + 2k+1 & \pmod{n}. \end{array}$$

We show that τ is a permutation. Note that $(n,k) \leq 2$ and $3k+2 \equiv -k \pmod{n}$. Thus, $(i-i')(3k+2) \equiv k(i'-i) \not\equiv 0 \pmod{n}$ if $0 < i - i' < n/2$. This implies that $\tau(2i) \neq \tau(2i')$ and $\tau(2i+1) \neq \tau(2i'+1)$ if $i \neq i'$.

If $\tau(2i) = \tau(2i'+1)$, then similarly, we get $(i-i')k \equiv 2k+1 = n/2 \pmod{n}$. Since $\gcd(n/2, k) = 1$ and $|i - i'| \leq 2k < n/2$, this is impossible.

The span of f is $f_0 + f_1 + \cdots + f_{n-2} = 2k(k+2)+1$.

$\boxed{\text{Case 3. } n = 4k+1}$ In this case, $d = 2k$. The distance gap sequence D is defined by:

$$d_{4i} = d_{4i+2} = 2k - i \quad \text{and} \quad d_{4i+1} = d_{4i+3} = k+1+i.$$

The color gap sequence F is defined by

$$f_i = d - d_i + 1 = 2k - d_i + 1.$$

Then, the mapping τ on the vertices of C_n has

$$\begin{array}{lcll} \tau(2i) & = & i(3k+1) \pmod{n} & \\ & = & -ik \pmod{n}, & 0 \leq i \leq 2k; \\ & & & \\ \tau(4i+1) & = & 2i(3k+1) + 2k - i \pmod{n} & \\ & = & 2(i+1)k \pmod{n}, & 0 \leq i \leq k-1; \\ & & & \\ \tau(4i+3) & = & (2i+1)(3k+1) + 2k - i \pmod{n} & \\ & = & (2i+1)k \pmod{n}, & 0 \leq i \leq k-1. \end{array}$$

We show that τ is indeed a permutation. Let

$$\begin{aligned} S &= \{-i : 0 \le i \le 2k\} \cup \{2(i+1) : 0 \le i \le k-1\} \\ &\quad \cup \{2i+1 : 0 \le i \le k-1\} \\ &= \{-2k, -(2k-1), \cdots, 0, 1, \cdots, 2k\}. \end{aligned}$$

By the definition of τ, for any $0 \le j \le 4k$ we have $\tau(j) = a_j k \pmod{n}$ for some $a_j \in S$, and $a_j \ne a_{j'}$, if $j \ne j'$. Thus to prove $\tau(j) \ne \tau(j')$ for $j \ne j'$, it suffices to show that for any distinct elements a, a' of S, $ak \ne a'k \pmod{n}$. This is obvious, as $\gcd(n,k) = 1 \pmod{n}$ and for any two distinct elements a, a' of S, $0 < |a - a'| < n$. So $(a-a')k \not\equiv 0 \pmod{n}$, and hence τ is a permutation.

Using the fact that $d_{2i} + d_{2i+1} = 3k+1$ for any i, the span of f is

$$\begin{aligned} f_0 + f_1 + f_2 + \cdots + f_{n-2} &= (4k)(2k) - (d_0 + d_1 + \cdots + d_{n-2}) + 4k \\ &= 8k^2 - 2k(3k+1) + 4k \\ &= 2k(k+1). \end{aligned}$$

$\boxed{\text{Case 4. } n = 4k+3}$ In this case, $d = 2k+1$. The distance gap sequence D is defined by

$$d_{4i} = d_{4i+2} = 2k+1-i, \ d_{4i+1} = k+1+i, \ d_{4i+3} = k+2+i.$$

The coloring gap sequence F is

$$f_i = \begin{cases} d - d_i + 1 = 2k - d_i + 2, & i \not\equiv 3 \pmod{4}; \\ d - d_i + 2 = 2k - d_i + 3, & \text{otherwise.} \end{cases}$$

Then the mapping τ on the vertices of C_n has

$$\begin{array}{rcll}
\tau(4i) & = & i(6k+5) \pmod{n} & \\
 & = & 2i(k+1) \pmod{n}, & 0 \le i \le k; \\
 & & & \\
\tau(4i+1) & = & 2i(k+1)+2k+1-i \pmod{n} & \\
 & = & (i+1)(2k+1) \pmod{n} & \\
 & = & -2(i+1)(k+1) \pmod{n}, & 0 \le i \le k; \\
 & & & \\
\tau(4i+2) & = & (i+1)(2k+1)+k+1+i \pmod{n} & \\
 & = & (i+1)(2k+2)+k \pmod{n} & \\
 & = & 2(i+1)(k+1)-3(k+1) \pmod{n} & \\
 & = & (2i-1)(k+1) \pmod{n}, & 0 \le i \le k; \\
 & & & \\
\tau(4i+3) & = & 2i(k+1)+3k+2+2k+1-i \pmod{n} & \\
 & = & i(2k+1)+k \pmod{n} & \\
 & = & -i(2k+2)-3(k+1) \pmod{n} & \\
 & = & -(2i+3)(k+1) \pmod{n}, & 0 \le i \le k-1.
\end{array}$$

Now we prove that τ is a permutation. Let

$$\begin{aligned}
S &= \{2i : 0 \le i \le k\} \cup \{-2(i+1) : 0 \le i \le k\} \\
&\quad \cup \{2i-1 : 0 \le i \le k\} \cup \{-(2i+3) : 0 \le i \le k-1\} \\
&= \{-(2k+2), -(2k+1), \cdots, 0, 1, \cdots, 2k\}.
\end{aligned}$$

By the definition of τ, for any $0 \le j \le 4k+2$, we have $\tau(j) = a_j(k+1) \pmod{n}$ for some $a_j \in S$, and $a_j \ne a_{j'}$ if $j \ne j'$. Thus, to prove $\tau(j) \ne \tau(j')$ for $j \ne j'$, it suffices to show that for any distinct elements a, a' of S, $a(k+1) \ne a'(k+1) \pmod{n}$. This is obvious, as $(n, k+1) = 1 \pmod{n}$ and for any two distinct elements a, a' of S, $0 < |a - a'| < n$. Hence, τ is a permutation.

The span of f is

$$f_0 + f_1 + \cdots + f_{n-2} = 2k(4k+2) - (d_0 + d_1 + \cdots + d_{n-2}) + 2(4k+2) + k$$

$$= \quad 2k(4k+2) - [k(6k+5) + 3k + 2] + 9k + 4$$
$$= \quad (k+2)(2k+1).$$

This completes the proof of Theorem 5. ∎

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References

[1] G. Chang, C. Ke, D. Kuo, D. Liu, and R. Yeh, *A generalized distance two labeling of graphs*, Disc. Math., 220 (2000), 57 – 66.

[2] G. Chang and D. Kuo, *The* $L(2,1)$*-labeling problem on graphs*, SIAM J. Disc. Math., 9 (1996), 309 – 316.

[3] G. Chartrand, D. Erwin, F. Harary and P. Zhang, *Radio labelings of graphs*, Bull. Inst. Combin. Appl., 33 (2001), 77 – 85.

[4] G. Chartrand, D. Erwin, and P. Zhang, *A graph labeling problem suggested by FM channel restrictions*, Bull. Inst. Combin. Appl., 43 (2005), 43–57.

[5] J. Georges and D. Mauro, *Generalized vertex labelings with a condition at distance two*, Congr. Numer., 109 (1995), 141 – 159.

[6] J. Georges, D. Mauro, and M. Stein, *Labeling products of complete graphs with a condition at distance two*, SIAM J. Discrete Math., 14 (2001), 28 – 35.

[7] J. Georges, D. Mauro, and M. Whittlesey, *On the size of graphs labeled with a condition at distance two*, J. Graph Theory, 22 (1996), 47 – 57.

[8] J. Georges, D. Mauro, and M. Whittlesey, *Relating path covering to vertex labelings with a condition at distance two*, Disc. Math., 135 (1994), 103 –111.

[9] J. R. Griggs and R. K. Yeh, *Labeling graphs with a condition at distance 2*, SIAM J. Disc. Math., 5 (1992), 586 – 595.

[10] W. K. Hale, *Frequency assignment: theory and applications*, Proc. IEEE, 68 (1980), 1497 – 1514.

[11] J. van den Heuvel, R. Leese, and M. Shepherd, *Graph labeling and radio channel assignment*, J. Graph Theory, 29 (1998), 263 – 283.

[12] D. Liu and R. K. Yeh, *On distance two labellings of graphs*, Ars Comb., 47 (1997), 13 – 22.

[13] D. Liu and X. Zhu, *Circular distance two labeling and the λ-number for outerplanar graphs*, SIAM J. Disc. Math., in press, 2005.

[14] D. Sakai, *Labeling chordal graphs with a condition at distance two*, SIAM J. Disc. Math., 7 (1994), 133 – 140.

[15] M. Whittlesey, J. Georges, and D. Mauro, *On the λ-number of Q_n and related graphs*, SIAM J. Disc. Math., 8 (1995), 499 – 506.