Study of $\kappa(D)$ for $D = \{2, 3, x, y\}$ *

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Abstract

Let D be a set of positive integers. The *kappa value* of D, denoted by $\kappa(D)$, is the parameter involved in the so called "lonely runner conjecture." Let x, y be positive integers, we investigate the kappa values for the family of sets $D = \{2, 3, x, y\}$. For a fixed positive integer $x > 3$, the exact values of $\kappa(D)$ are determined for $y = x + i$, $1 \le i \le 6$. These results lead to some asymptotic behavior of $\kappa(D)$ for $D = \{2, 3, x, y\}$.

1 Introduction

Let D be a set of positive integers. For any real number x, let $||x||$ denote the minimum distance from x to an integer, that is, $||x|| = \min\{\lceil x \rceil - x, x - \lfloor x \rfloor\}$. For any real t, denote $||tD||$ the smallest value $||td||$ among all $d \in D$. The *kappa value of* D, denoted by $\kappa(D)$, is the supremum of $||tD||$ among all real t. That is,

$$\kappa(D) := \sup\{\alpha : ||tD|| \ge \alpha \text{ for some } t \in \Re\}.$$

Wills [20] conjectured that $\kappa(D) \ge 1/(|D|+1)$ is true for all finite sets D. This conjecture is also known as the *lonely runner conjecture* by Bienia et al. [2]. Suppose m runners run laps on a circular track of unit circumference.

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Each runner maintains a constant speed, and the speeds of all the runners are distinct. A runner is called *lonely* if the distance on the circular track between him or her and every other runner is at least $1/m$. Equivalently, the conjecture asserts that for each runner, there is some time t when he or she becomes lonely. The conjecture has been proved true for $|D| \leq 6$ (cf. [1, 3, 6, 7]), and remains open for $|D| \geq 7$.

The parameter $\kappa(D)$ is closely related to another parameter of D called the "density of integral sequences with missing differences." For a set D of positive integers, a sequence S of non-negative integers is called a *D-sequence* if $|x-y| \notin D$ for any $x, y \in S$. Denote $S(n)$ as $|S \cap \{0, 1, 2, \cdots, n-1\}|$. The upper density $\overline{\delta}(S)$ and the lower density $\underline{\delta}(S)$ of S are defined, respectively, by $\overline{\delta}(S) = \overline{\lim}_{n\to\infty} S(n)/n$ and $\underline{\delta}(S) = \underline{\lim}_{n\to\infty} S(n)/n$. We say S has density $\delta(S)$ if $\overline{\delta}(S) = \underline{\delta}(S) = \delta(S)$. The parameter of interest is the *density of D*, $\mu(D)$, defined by

$$\mu(D) := \sup \{\ \delta(S) :\ S \text{ is a } D\text{-sequence}\}.$$

It is known that for any set D (cf. [4]):

$$\mu(D) \geq \kappa(D). \tag{1}$$

For two-element sets $D = \{a, b\}$, Cantor and Gordon [4] proved that $\kappa(D) = \mu(D) = \frac{\lfloor \frac{a+b}{2} \rfloor}{a+b}$. For 3-element sets D, if $D = \{a, b, a+b\}$ it was proved that $\kappa(D) = \mu(D)$ and the exact values were determined (see Theorem 2 below). For the general case $D = \{i, j, k\}$, various lower bounds of $\kappa(D)$ were given by Gupta [11], in which the values of $\mu(D)$ were also studied. In addition, among other results it was shown in [11] that if D is an arithmetic sequence then $\kappa(D) = \mu(D)$ and the value was determined.

The parameters $\kappa(D)$ and $\mu(D)$ are closely related to coloring parameters of distance graphs. Let D be a set of positive integers. The *distance graph generated by D*, denoted as $G(\mathbb{Z}, D)$, has all integers $\mathbb{Z}$ as the vertex set. Two vertices are adjacent whenever their absolute value difference falls in D. The *chromatic number* (minimum number of colors in a proper vertex-coloring) of the distance graph generated by D is denoted by $\chi(D)$. It is known that $\chi(D) \leq \lceil 1/\kappa(D) \rceil$ for any set D (cf. [21]).

The *fractional chromatic number* of a graph G, denoted by $\chi_f(G)$, is the minimum ratio m/n ($m, n \in \mathbb{Z}^+$) of an (m/n)-coloring, where an (m/n)-coloring is a function on $V(G)$ to n-element subsets of $[m] = \{1, 2, \cdots, m\}$

such that if $uv \in E(G)$ then $f(u) \cap f(v) = \emptyset$. It is known that for any graph G, $\chi_f(G) \leq \chi(G)$ (cf. [21]).

Denote the fractional chromatic number of $G(\mathbb{Z}, D)$ by $\chi_f(D)$. Chang et al. [5] proved that for any set of positive integers D, it holds that $\chi_f(D) = 1/\mu(D)$. Together with (1) we obtain

$$\frac{1}{\mu(D)} = \chi_f(D) \leq \chi(D) \leq \lceil \frac{1}{\kappa(D)} \rceil. \tag{2}$$

The chromatic number of distance graphs $G(\mathbb{Z}, D)$ with $D = \{2, 3, x, y\}$ was studied by several authors. For prime numbers x and y, the values of $\chi(D)$ for this family were first studied by Eggleton, Erdős and Skilton [10] and later on completely solved by Voigt and Walther [18]. For general values of x and y, Kemnitz and Kolberg [13] and Voigt and Walther [19] determined $\chi(D)$ for some values of x and y. This problem was completely solved for all values of x and y by Liu and Setudja [15], in which $\kappa(D)$ was utilized as one of the main tools. In particular, it was proved in [15] that $\kappa(D) \geq 1/3$ for many sets in the form $D = \{2, 3, x, y\}$. By (2), for those sets it holds that $\chi(D) = 3$.

In this article we further investigate those previously established lower bounds of $\kappa(D)$ for the family of sets $D = \{2, 3, x, y\}$. In particular, we determine the exact values of $\kappa(D)$ for $D = \{2, 3, x, y\}$ with $|x - y| \leq 6$. Furthermore, for some cases it holds that $\kappa(D) = \mu(D)$. Our results also lead to asymptotic behavior of $\kappa(D)$.

2 Preliminaries

We introduce terminologies and known results that will be used to determine the exact values of $\kappa(D)$. It is easy to see that if the elements of D have a common factor r, then $\kappa(D) = \kappa(D')$ and $\mu(D) = \mu(D')$, where $D' = D/r = \{d/r : d \in D\}$. Thus, throughout the article we assume that $\gcd(D) = 1$, unless it is indicated otherwise.

The following proposition is derived directly from definitions.

Proposition 1. *If $D \subseteq D'$ then $\kappa(D) \geq \kappa(D')$ and $\mu(D) \geq \mu(D')$.*

The next result was established by Liu and Zhu [16], after confirming a conjecture of Rabinowitz and Proulx [17].

Theorem 2. [16] *Suppose $M = \{a, b, a+b\}$ for some positive integers a and b with $gcd(a,b) = 1$. Then*

$$\mu(M) = \kappa(M) = \max\left\{ \frac{\lfloor \frac{2b+a}{3} \rfloor}{2b+a}, \frac{\lfloor \frac{2a+b}{3} \rfloor}{2a+b} \right\}.$$

By Proposition 1, if $\{a, b, a+b\} \subseteq D$ for some a and b, then Theorem 2 gives an upper bound for $\kappa(D)$.

For a D-sequence S, denote $S[n] = |\{0, 1, 2, \ldots, n\} \cap S|$. The next result was proved by Haralambis [12].

Lemma 3. [12] *Let D be a set of positive integers, and let $\alpha \in (0, 1]$. If for every D-sequence S with $0 \in S$ there exists a positive integer n such that $\frac{S[n]}{n+1} \leq \alpha$, then $\mu(D) \leq \alpha$.*

For a given D-sequence S, we shall write elements of S in an increasing order, $S = \{s_0, s_1, s_2, \ldots\}$ with $s_0 < s_1 < s_2 < \ldots$, and denote its *difference sequence* by

$$\Delta(S) = \{\delta_0, \delta_1, \delta_2, \ldots\} \text{ where } \delta_i = s_{i+1} - s_i.$$

We call a subsequence of consecutive terms in $\Delta(S)$, $\delta_a, \delta_{a+1}, \ldots, \delta_{a+b-1}$, generates a periodic interval of k copies, $k \geq 1$, if $\delta_{j(a+b)+i} = \delta_{a+i}$ for all $0 \leq i \leq b-1$, $1 \leq j \leq k-1$. We denote such a periodic subsequence of $\Delta(S)$ by $(\delta_a, \delta_{a+1}, \ldots, \delta_{a+b-1})^k$. If the periodic interval repeats infinitely, then we simply denote it by $(\delta_a, \delta_{a+1}, \ldots, \delta_{a+b-1})$. If $\Delta(S)$ is infinite periodic, except the first finite number of terms, with the periodic interval $(t_1, t_2, \ldots, t_k)$, then the density of S is $k/(\sum_{i=1}^{k} t_i)$.

Proposition 4. *A sequence of non-negative integers S is a D-sequence if and only if $\sum_{i=a}^{b} \delta_i \notin D$ for every $a \leq b$.*

Proposition 5. *Assume $2, 3 \in D$. If S is a D-sequence, then $\delta_i + \delta_{i+1} \geq 5$ for all i. The equality holds only when $\{\delta_i, \delta_{i+1}\} = \{1, 4\}$. Consequently, $\mu(D) \leq 2/5$.*

Lemma 6. *Let $D = \{2, 3\} \cup A$. Then $\kappa(D) = 2/5$ if and only if $A \subseteq \{x : x \equiv 2, 3 \pmod 5\}$. Furthermore, if $\kappa(D) = 2/5$, then $\mu(D) = 2/5$.*

Proof. Let $D = \{2,3\} \cup A$. Assume $A \subseteq \{x : x \equiv 2,3 \pmod 5\}$. Let $t = 1/5$. Then $||td|| \geq 2/5$ for all $d \in D$. Hence $\kappa(D) \geq 2/5$. On the other hand, the density of the infinite periodic D-sequence S with $\Delta(S) = (1,4)$ is $2/5$. By Proposition 5, this is an optimal D-sequence. Hence, $\mu(D) = 2/5$, implying $\kappa(D) = 2/5$.

Conversely, assume $\kappa(D) = 2/5$. Then $\mu(D) \geq 2/5$. By Proposition 5, $\mu(D) = 2/5$. By Proposition 4, this implies that if $d \in D$, then $d \not\equiv 0,1,4 \pmod 5$. Thus the result follows. □ □

Note, in $D = \{2,3,x,y\}$, if $x = 1$, then it is known [16] and easy to see that $\mu(D) = \kappa(D) = 1/4$ if y is not a multiple of 4 (with $\Delta(S) = (4)$); otherwise $y = 4k$ and $\mu(D) = \kappa(D) = k/(4k+1)$ (with $\Delta(S) = ((4)^{k-1}5)$). Hence throughout the article we assume $x > 3$.

Another method we will utilize is an alternative definition of $\kappa(D)$. In this definition, for a projected lower bound α of $\kappa(D)$, for each element z in D the valid *time* t for z to achieve α is expressed as a union of disjoint intervals. Let $\alpha \in (0, \frac{1}{2})$. For positive integer i, define $I_i(\alpha) = \{t \in (0,1) : \| \, ti \, \| \geq \alpha\}$. Equivalently,

$$I_i(\alpha) = \{t : n + \alpha \leq ti \leq n + 1 - \alpha, 0 \leq n \leq i-1\}.$$

That is, I_i consists of intervals of reals with length $(1-2\alpha)/i$ and centered at $(2n+1)/(2i)$, $n = 0,1,\ldots,i-1$. By definition, $\kappa(D) \geq \alpha$ if and only if $\bigcap_{i \in D} I_i(\alpha) \neq \emptyset$. Thus,

$$\kappa(D) = \sup \left\{ \alpha \in (0, \frac{1}{2}) : \bigcap_{i \in D} I_i(\alpha) \neq \emptyset \right\}.$$

Observe that if $\bigcap\limits_{i \in D} I_i(\alpha)$ consists of only isolated points, then $\kappa(D) \leq \alpha$. Hence, we have the following:

Proposition 7. *For a set D, $\kappa(D) \leq d/c$ if $\bigcap\limits_{i \in D} I_i$ is a set of isolated points, where*

$$I_i = \bigcup_{n=0}^{i-1} \left[\frac{d+cn}{i}, \frac{c-d+cn}{i} \right].$$

3 $D = \{2, 3, x, y\}$ for $y = x + 1, x + 2, x + 3$

Theorem 8. *Let $D = \{2, 3, x, x + 1\}$, $x \geq 4$. Then*

$$\kappa(D) = \mu(D) = \begin{cases} \frac{2\lfloor \frac{x+3}{5} \rfloor + 1}{x+3} & \text{if } x \equiv 1 \text{ (mod 5);} \\ \frac{2\lfloor \frac{x+3}{5} \rfloor}{x+3} & \text{otherwise.} \end{cases}$$

Proof. We prove the following cases.

Case 1. $\boldsymbol{x = 5k + 2}$. The result follows by Lemma 6.

Case 2. $\boldsymbol{x = 5k+3}$. Let $t = (k+1)/(5k+6)$. Then $||dt|| \geq (2k+2)/(5k+6)$ for every $d \in D$. Hence $\kappa(D) \geq (2k + 2)/(5k + 6)$.

By (1) it remains to show that $\mu(D) \leq (2k + 2)/(5k + 6)$. Assume to the contrary that $\mu(D) > (2k + 2)/(5k + 6)$. By Lemma 3, there exists a D-sequence S with $S[n]/(n + 1) > (2k + 2)/(5k + 6)$ for all $n \geq 0$. This implies, for instance, $S[0] \geq 1$, so $s_0 = 0$; $S[2] \geq 2$, so $s_1 = 1$ (as $2, 3 \in D$); $S[5] \geq 3$, so $s_3 = 5$. Moreover, $S[5k + 5] \geq 2k + 3$. By Proposition 5, it must be $(\delta_0, \delta_1, \delta_2, \ldots, \delta_{2k+1}) = (1, 4, 1, 4, \ldots, 1, 4)$. This implies $5k + 5 \in S$, which is impossible since $1 \in S$ and $5k + 4 \in D$. Therefore, $\mu(D) = \kappa(D) = (2k + 2)/(5k + 6)$.

Case 3. $\boldsymbol{x = 5k+4}$. Let $t = (k+1)/(5k+7)$. Then $||dt|| \geq (2k+2)/(5k+7)$ for all $d \in D$. Hence $\kappa(D) \geq (2k + 2)/(5k + 7)$.

By (1) it remains to show that $\mu(D) \leq (2k + 2)/(5k + 7)$. Assume to the contrary that $\mu(D) > (2k + 2)/(5k + 7)$. By Lemma 3, there exists a D-sequence S with $S[n]/(n + 1) > (2k + 2)/(5k + 7)$ for all $n \geq 0$. This implies, for instance, $S[0] \geq 1$, so $s_0 = 0$; $S[3] \geq 2$, so $s_1 = 1$ (as $2, 3 \in D$); and $S[5k + 6] \geq 2k + 3$. By Proposition 5, either $5k + 5$ or $5k + 6$ is an element in S. This is impossible since $0, 1 \in S$ and $5k + 4, 5k + 5 \in D$. Thus $\mu(D) = \kappa(D) = (2k + 2)/(5k + 7)$.

Case 4. $\boldsymbol{x = 5k+5}$. Let $t = (k+1)/(5k+8)$. Then $||dt|| \geq (2k+2)/(5k+8)$ for all $d \in D$. Hence $\kappa(D) \geq (2k + 2)/(5k + 8)$.

It remains to show $\mu(D) \leq (2k + 2)/(5k + 8)$. Assume to the contrary that $\mu(D) > (2k + 2)/(5k + 8)$. By Lemma 3, there exists a D-sequence S with $S[n]/(n + 1) > (2k + 2)/(5k + 8)$ for all $n \geq 0$. Similar to the above, one has $0, 1 \in S$ and $S[5k + 7] \geq 2k + 3$. This implies that one of $5k + 5$, $5k + 6$, or $5k + 7$ is an element in S, which is again impossible. Therefore, $\mu(D) = \kappa(D) = (2k + 2)/(5k + 8)$.

Case 5. $x = 5k+1$. Let $t = (k+1)/(5k+4)$. Then $||dt|| \geq (2k+1)/(5k+4)$ for all $d \in D$. Hence $\kappa(D) \geq (2k+1)/(5k+4)$.

Now we show $\mu(D) \leq (2k+1)/(5k+4)$. Assume to the contrary that $\mu(D) > (2k+1)/(5k+4)$. By Lemma 3, $(s_0, s_1) = (0, 1)$, and $S[5k+3] \geq 2k+2$. Because $S[5k] \leq 2k+1$, so $S \cap \{5k+1, 5k+2, 5k+3\} \neq \emptyset$, which is impossible. Therefore, $\mu(D) = \kappa(D) = (2k+1)/(5k+4)$. □ □

By the above proofs, one can extend the family of sets D to the following:

Corollary 9. *Let $D = \{2, 3, x, x+1\} \cup D'$, where $D' \subseteq \{y : y \equiv \pm 2, \pm 3 \pmod{(x+3)}\}$. Then $\mu(D) = \kappa(D) = \mu(\{2, 3, x, x+1\})$.*

Corollary 10. *Let $D = \{2, 3, x, x+1\}$. Then*

$$\lim_{x \to \infty} \kappa(D) = \frac{2}{5}.$$

Theorem 11. *Let $D = \{2, 3, x, x+2\}$, $x \geq 4$. Assume $x + 4 = 6\beta + r$ with $0 \leq r \leq 5$. Then*

$$\kappa(D) = \begin{cases} \frac{\lfloor \frac{x+4}{3} \rfloor}{x+4} & \text{if } 0 \leq r \leq 2; \\ \\ \frac{\lfloor \frac{2x+1}{3} \rfloor}{2x+2} & \text{if } 3 \leq r \leq 5. \end{cases}$$

Furthermore, $\kappa(D) = \mu(D)$ if $r \neq 3$.

Proof. We prove the following cases.

Case 1. $x = 6k+2$. Then $r = 0$. Let $t = 1/6$. Then $||dt|| \geq 1/3$ for all $d \in D$. Hence $\kappa(D) \geq 1/3$.

Now we prove $\mu(D) \leq 1/3$. Let $M' = \{2, x, x+2\} = \{2, 6k+2, 6k+4\}$. By Theorem 2 with $M = \{1, 3k+1, 3k+2\}$, we obtain $\mu(M') = \mu(M) = 1/3$. Because $M' \subseteq D$, so $\mu(D) \leq \mu(M') = 1/3$.

Case 2. $x = 6k+3$. Then $r = 1$. Let $t = (k+1)/(6k+7)$. Then $||dt|| \geq (2k+2)/(6k+7)$ for all $d \in D$. Hence $\kappa(D) \geq (2k+2)/(6k+7)$.

By Theorem 2 with $M = \{2, x, x+2\} = \{2, 6k+3, 6k+5\}$, we get $\mu(M) = (2k+2)/(6k+7)$. Because $M \subseteq D$, so $\mu(D) \leq \mu(M) = (2k+2)/(6k+7)$. Thus, the result follows.

Case 3. $x = 6k+4$. Then $r = 2$. Let $t = (k+1)/(6k+8)$. Then $||dt|| \geq (2k+2)/(6k+8)$ for all $d \in D$. Hence $\kappa(D) \geq (2k+2)/(6k+8)$.

By Theorem 2 with $M = \{2, x, x+2\} = \{2, 6k+4, 6k+6\}$ which can be reduced to $M' = \{1, 3k+2, 3k+3\}$, we obtain $\mu(M) = (k+1)/(3k+4)$. Therefore, $\mu(D) \leq \mu(M) = (2k+2)/(6k+8)$. So the result follows.

Case 4. $\boldsymbol{x = 6k+5}$. Then $r = 3$. Let $t = (2k+3)/(12k+12)$. Then $||dt|| \geq (4k+3)/(12k+12)$ for all $d \in D$. Hence $\kappa(D) \geq (4k+3)/(12k+12)$.

By Proposition 7, it remains to show that $\bigcap_{i=2,3,x,x+2} I_i$ is a set of isolated points, where

$$I_i = \bigcup_{n=0}^{i-1} \left[\frac{4k+3+n(12k+12)}{i}, \frac{8k+9+n(12k+12)}{i} \right].$$

Let $I = \bigcap_{i=2,3,x,x+2} I_i$. By symmetry it is enough to consider the interval $I \cap [0, (12k+12)/2]$. In the following we claim $I \cap [0, 6k+6] = \{2k+3\}$. (Indeed, this single point is the numerator of the t value at the beginning of the proof.)

Note that $I_2 \cap I_3 \cap [0, 6k+6] = [(4k+3)/2, (8k+9)/3]$. Denote this interval by

$$I_{2,3} = \left[\frac{4k+3}{2}, \frac{8k+9}{3} \right].$$

We then begin to investigate possible values of n for I_x and I_{x+2}, respectively, that will fall within $I_{2,3}$. First, we compare the I_x intervals with $I_{2,3}$. Recall

$$I_x = \left[\frac{3+4k+n(12+12k)}{6k+5}, \frac{8k+9+n(12+12k)}{6k+5} \right], \ 0 \leq n \leq 6k+4.$$

By calculation, the intervals of I_x that intersect with $I_{2,3}$ are those with $n \geq k$. Similarly, we compare I_{x+2} intervals with $I_{2,3}$. Recall

$$I_{x+2} = \left[\frac{3+4k+n(12+12k)}{6k+7}, \frac{8k+9+n(12+12k)}{6k+7} \right], \ 0 \leq n \leq 6k+6.$$

By calculation, the intervals of I_{x+2} that intersect with $I_{2,3}$ are those with $n \geq k+1$.

Next, we consider the intersection between intervals of I_x and I_{x+2}. Let $n = k+a$ for some $a \geq 0$ for the I_x interval, and let $n = k+a'$ for some $a' \geq 1$ for the I_{x+2} interval. By taking the common denominator of the I_x and I_{x+2} intervals we obtain the following numerators of those intervals:

$$\text{for } I_x : [21 + 84a + 130k + 156ak + 180k^2 + 72ak^2 + 72k^3,$$

$$63 + 84a + 194k + 156ak + 204k^2 + 72ak^2 + 72k^3];$$

$$\text{for } I_{x+2} : [15 + 60a' + 98k + 132a'k + 156k^2 + 72a'k^2 + 72k^3,$$

$$45 + 60a' + 154k + 132a'k + 180k^2 + 72a'k^2 + 72k^3].$$

Using $a = a' = 1$, we get

$$\text{for } I_x : [105 + 286k + 252k^2 + 72k^3, 147 + 350k + 276k^2 + 72k^3]$$

$$\text{for } I_{x+2} : [75 + 230k + 228k^2 + 72k^3, 105 + 286k + 252k^2 + 72k^3].$$

Thus, there is a single point intersection for I_x and I_{x+2} when $a = a' = 1$, which is $\{2k+3\}$. This single point intersection is also within the $I_{2,3}$ interval. Hence, $\{2k + 3\} \in I \cap [0, 6k + 6]$.

In addition, through inspection it is clear that making $n = k$ (i.e. $a = 0$) for the I_x interval and $n \geq k+1$ ($a' \geq 1$) for the I_{x+2} interval removes I_x and I_{x+2} from intersecting one another. For all other cases, $a = 1$ and $a' \geq 2$, $a \geq 2$ and $a' = 1$, or $a, a' \geq 2$, there will never be an intersection of intervals for all elements in D, either because the $I_{2,3}$ interval is too small or because the I_{x+2} elements become too big. Thus, $I \cap [0, 6k + 6] = \{2k + 3\}$.

Case 5. $\boldsymbol{x = 6k + 6}$**.** Then $r = 4$. Let $t = (2k + 3)/(12k + 14)$. Then $||dt|| \geq (4k+4)/(12k+14)$ for all $d \in D$. Hence $\kappa(D) \geq (4k+4)/(12k+14)$.

By Theorem 2 with $M = \{2, x, x + 2\} = \{2, 6k + 6, 6k + 8\}$ which can be reduced to $M' = \{1, 3k+3, 3k+4\}$, we get $\mu(M) = \kappa(M) = (2k+2)/(6k+7)$. Hence, $\mu(D) \leq \mu(M) = (2k + 2)/(6k + 7)$.

Case 6. $\boldsymbol{x = 6k + 7}$**.** Then $r = 5$. Let $t = (2k + 3)/(12k + 16)$. Then $||dt|| \geq (4k+5)/(12k+16)$ for all $d \in D$. Hence $\kappa(D) \geq (4k+5)/(12k+16)$.

By Theorem 2 with $M = \{2, x, x + 2\} = \{2, 6k + 7, 6k + 9\}$, we obtain $\mu(M) = \kappa(M) = (4k + 5)/(12k + 16)$. Therefore, $\mu(D) \leq (4k + 5)/(12k + 16)$. □ □

Theorem 12. *Let* $D = \{2, 3, x, x + 3\}$*,* $x \geq 4$*. Assume* $(2x + 3) = 9\beta + r$ *with* $0 \leq r \leq 8$*. Then*

$$\kappa(D) = \begin{cases} \frac{3\lfloor \frac{2x+3}{9} \rfloor}{2x+3} & \text{if } 0 \leq r \leq 5; \\ \frac{\lfloor \frac{x+5}{3} \rfloor}{x+6} & \text{if } 6 \leq r \leq 8. \end{cases}$$

Furthermore, if $r = 0, 1, 3, 6, 8$ *then* $\kappa(D) = \mu(D)$.

Proof. We prove the following cases:

Case 1. $\boldsymbol{x = 9k + 3}$**.** Then $r = 0$. Let $t = 2/9$. Then $||dt|| \geq 1/3$ for all $d \in D$. Hence $\kappa(D) \geq (6k+3)/(18k+9) = 1/3$.

By Theorem 2 with $M = \{3, x, x+3\} = \{3, 9k+3, 9k+6\}$, which can be reduced to $M' = \{1, 3k+1, 3k+2\}$, resulting in $\mu(M) = \kappa(M) = 1/3$. Because $M \subseteq D$, so $\mu(D) = \mu(M) = 1/3$.

Case 2. $\boldsymbol{x = 9k + 8}$**.** Then $r = 1$. Let $t = (4k+4)/(18k+19)$. Then $||dt|| \geq (6k+6)/(18k+19)$ for all $d \in D$. Hence $\kappa(D) \geq (6k+6)/(18k+19)$.

By Theorem 2 with $M = \{3, x, x+3\}$, we get $\kappa(M) = (6k+6)/(18k+19)$. Hence, $\mu(D) \leq \kappa(M) = (6k+6)/(18k+19)$.

Case 3. $\boldsymbol{x = 9k + 4}$**.** Then $r = 2$. Let $t = (4k+2)/(18k+11)$. Then $||dt|| \geq (6k+3)/(18k+11)$ for all $d \in D$. Thus, $\kappa(D) \geq (6k+3)/(18k+11)$.

The proof for the other direction is similar to the proof of Case 4 in Theorem 11. Let $I = \bigcap_{i=2,3,x,x+3} I_i$. By calculation we have $I \cap [0, 9k+(11/2)] = \{4k+2\}$. This single point of intersection occurs when $n = 2k$ in the I_x interval, and $n = 2k+1$ in the I_{x+3} interval.

Case 4. $\boldsymbol{x = 9k}$**.** Then $r = 3$. Let $t = 4k/(18k+3)$. Then $||dt|| \geq (6k)/(18k+3)$ for all $d \in D$. Thus $\kappa(D) \geq (2k)/(6k+1)$.

By Theorem 2 with $M = \{3, x, x+3\} = \{3, 9k, 9k+3\}$, $\mu(M) = \kappa(M) = (2k)/(6k+1)$. Hence, the result follows.

Case 5. $\boldsymbol{x = 9k + 5}$**.** Then $r = 4$. Let $t = (4k+2)/(18k+13)$. Then $||dt|| \geq (6k+3)/(18k+13)$ for all $d \in D$. Thus $\kappa(D) \geq (6k+3)/(18k+13)$.

The proof for the other direction is similar to the proof of Case 4 in Theorem 11. Let $I = \bigcap_{i=2,3,x,x+3} I_i$. By calculation we have $I \cap [0, 9k+(13/2)] = \{4k+2\}$. This single point of intersection occurs when $n = 2k$ in the I_x interval, and $n = 2k+1$ in the I_{x+3} interval.

Case 6. $\boldsymbol{x = 9k + 1}$**.** Then $r = 5$. Let $t = (4k)/(18k+5)$. Then $||dt|| \geq (6k)/(18k+5)$ for all $d \in D$. Thus $\kappa(D) \geq (6k)/(18k+5)$.

The proof for $\kappa(D) \leq (6k)/(18k+5)$ is similar to the proof of Case 4 in Theorem 11. Let $I = \bigcap_{i=2,3,x,x+3} I_i$. By calculation we have $I \cap [0, 9k+(5/2)] = \{4k\}$. This single point of intersection occurs when $n = 2k-1$ in the I_x interval, and $n = 2k$ in the I_{x+3} interval.

Case 7. $\boldsymbol{x = 9k + 6}$**.** Then $r = 6$. Let $t = (2k+3)/(9k+12)$. Then $||dt|| \geq (3k+3)/(9k+12)$ for all $d \in D$. Thus $\kappa(D) \geq (k+1)/(3k+4)$.

By Theorem 2 with $M = \{3, x, x+3\}$ with $M = \{3, x, x+3\} = \{3, 9t+6, 9t+9\}$, which can be reduced to $M' = \{1, 3t+2, 3t+3\}$, we get $\mu(M) = \kappa(M) = (k+1)/(3k+4)$. Because $M \subseteq D$, so $\kappa(D) \le \mu(D) \le \mu(M) \le \kappa(M) = (k+1)/(3k+4)$.

Case 8. $\boldsymbol{x = 9k+11}$**.** Then $r = 7$. Let $t = (2k+4)/(9k+17)$. Then $||dt|| \ge (3k+5)/(9k+17)$ for all $d \in D$. Thus $\kappa(D) \ge (3k+5)/(9k+17)$.

The proof for the other direction is similar to the proof of Case 4 in Theorem 11. Let $I = \bigcap_{i=2,3,x,x+3} I_i$. By calculation we have $I \cap [0, 9k+(17/2)] = \{2k+4\}$. This single point of intersection occurs when $n = 2k+2$ in the I_x interval, and $n = 2k+3$ in the I_{x+3} interval.

Case 9. $\boldsymbol{x = 9k+7}$**.** Then $r = 8$. Let $t = (2k+3)/(9k+13)$. Then $||dt|| \ge (3k+4)/(9k+13)$ for all $d \in D$. Thus $\kappa(D) \ge (3k+4)/(9k+13)$.

By Theorem 2 with $M = \{3, x, x+3\} = \{3, 9t+7, 9t+10\}$, we get $\kappa(M) = (3k+4)/(9k+13)$. Because $M \subseteq D$, so $\kappa(D) \le \mu(D) \le \mu(M) = \kappa(M) = (3k+4)/(9k+13)$. □ □

Corollary 13. *Let* $D = \{2, 3, x, y\}$ *where* $y \in \{x+2, x+3\}$*. Then*

$$\lim_{x \to \infty} \kappa(D) = \frac{1}{3}.$$

4 $D = \{2, 3, x, y\}$ for $y = x+4, x+5, x+6$

By similar proofs to the previous section, we obtain the following results.

Theorem 14. *Let* $D = \{2, 3, x, x+4\}$*,* $x \ge 4$*. Assume* $(x+4) = 5\beta + r$ *with* $0 \le r \le 4$*. Then*

$$\kappa(D) = \begin{cases} \frac{2\beta+r}{x+7} & \text{if } 0 \le r \le 1; \\ \mu(D) = \frac{2}{5} & \text{if } r = 2; \\ \frac{2\beta}{x+2} & \text{if } 3 \le r \le 4. \end{cases}$$

Proof. The case for $r = 2$ is from Lemma 6. The following table gives the corresponding t, $\kappa(D)$, and the n values of I_x and I_{x+4} where the single intersection point occurs.

x	r	t	n in I_x	n in I_{x+4}	$\kappa(D)$
$5k+4$	3	$(k+1)/(5k+6)$	k	$k+1$	$(2k+2)/(5k+6)$
$5k+5$	4	$(k+1)/(5k+7)$	k	$k+1$	$(2k+2)/(5k+7)$
$5k+6$	0	$(k+3)/(5k+13)$	$k+1$	$k+2$	$(2k+4)/(5k+13)$
$5k+7$	1	$(k+3)/(5k+14)$	$k+1$	$k+2$	$(2k+5)/(5k+14)$
$5k+8$	2	$1/5$			$2/5$

□ □

Theorem 15. *Let* $D = \{2, 3, x, x+5\}$, $x \geq 4$. *Assume* $(x+3) = 5\beta + r$ *with* $0 \leq r \leq 4$. *Then*

$$\kappa(D) = \begin{cases} \mu(D) = \frac{2}{5} & \text{if } 0 \leq r \leq 1; \\ \frac{2\beta}{x+2} & \text{if } 2 \leq r \leq 3; \\ \frac{2\beta+1}{x+3} & \text{if } r = 4. \end{cases}$$

Proof. The cases for $r = 0, 1$ are by Lemma 6. The following table gives the corresponding t, $\kappa(D)$, and the n values of I_x and I_{x+5} where the single intersection point occurs.

x	r	t	n in I_x	n in I_{x+5}	$\kappa(D)$
$5k+4$	2	$(k+1)/(5k+6)$	$k+1$	$k+1$	$(2k+2)/(5k+6)$
$5k+5$	3	$(k+1)/(5k+7)$	$k+1$	$k+1$	$(2k+2)/(5k+7)$
$5k+6$	4	$(k+2)/(5k+9)$	$k+1$	$k+2$	$(2k+3)/(5k+9)$
$5k+7$	0	$1/5$			$2/5$
$5k+8$	1	$1/5$			$2/5$

□ □

Theorem 16. *Let* $D = \{2, 3, x, x+6\}$, $x \geq 4$. *Assume* $(x+8) = 5\beta + r$ *with* $0 \leq r \leq 4$. *Then*

$$\kappa(D) = \begin{cases} \mu(D) = \frac{2}{7} & \text{if } x = 5; \\ \mu(D) = \frac{2}{5} & \text{if } r = 0; \\ \frac{2\beta}{x+8} & \text{if } 1 \leq r \leq 3 \text{ and } x \neq 5; \\ \frac{2\beta+1}{x+3} & \text{if } r = 4. \end{cases}$$

Proof. Assume $x = 5$. That is $D = \{2, 3, 5, 11\}$. Letting $t = 1/7$ we get $||td|| \geq 2/7$ for every $d \in D$. Hence, $\kappa(D) \geq 2/7$. On the other hand, by Theorem 2, $\mu(\{2, 3, 5\}) = 2/7$. Therefore, by (2), we have $\kappa(D) \leq \mu(D) \leq 2/7$.

The case for $r = 0$ is from Lemma 6. The following table gives the corresponding t, $\kappa(D)$, and the n values of I_x and I_{x+6} where the single intersection point occurs.

x	r	t	n in I_x	n in I_{x+6}	$\kappa(D)$
$5k+4$	2	$(k+2)/(5k+12)$	k	$k+1$	$(2k+4)/(5k+12)$
$5k+5$	3	$(k+2)/(5k+13)$	k	$k+1$	$(2k+4)/(5k+13)$
$5k+6$	4	$(k+2)/(5k+9)$	$k+1$	$k+2$	$(2k+3)/(5k+9)$
$5k+7$	0	$1/5$			$2/5$
$5k+8$	1	$(k+3)/(5k+16)$	$k+1$	$k+2$	$(2k+6)/(5k+16)$

□ □

Corollary 17. *Let $D = \{2, 3, x, y\}$ where $y \in \{x+4, x+5, x+6\}$. Then*

$$\lim_{x \to \infty} \kappa(D) = \frac{2}{5}.$$

Concluding remark and future study. Similar to Corollary 9, one can obtain sets D' that are extensions of the sets D studied in this article, $D \subset D'$, such that $\kappa(D) = \kappa(D')$. In addition, the methods used in this article can be applied to other sets $D = \{2, 3, x, x+c\}$ with $c \geq 7$. For a fixed c, preliminary results we obtained thus far indicate that the values of $\kappa(D)$ might be inconsistent for the first finite terms, while after a certain threshold, they seem to be more consistent (that is, most likely it can be described by a single formula). Thus, we would like to investigate whether the conclusion of Corollary 17 holds for all $D = \{2, 3, x, y\}$, $x < y$, where $y \neq x+2, x+3$? In a broader sense, it is interesting to further study the asymptotic behavior of $\kappa(D)$ for sets D containing 2 and 3, and identify any dominating factors for such behavior.

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