# A Combinatorial Proof for the Circular Chromatic Number of Kneser Graphs

Daphne Der-Fen Liu *
Department of Mathematics
California State University, Los Angeles, USA
Email: dliu@calstatela.edu

Xuding Zhu†
Department of Mathematics
Zhejiang Normal University, China
Email: xudingzhu@gmail.com

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**Abstract**

Chen [4] confirmed the Johnson-Holroyd-Stahl conjecture that the circular chromatic number of a Kneser graph is equal to its chromatic number. A shorter proof of this result was given by Chang, Liu, and Zhu [3]. Both proofs were based on Fan's lemma [5] in algebraic topology. In this article we give a further simplified proof of this result. Moreover, by specializing a constructive proof of Fan's lemma by Prescott and Su [19], our proof is self-contained and combinatorial.

## 1 Introduction

Let $G$ be a graph and $t$ a positive integer. A *proper* $t$-coloring of $G$ is a mapping that assigns to each vertex a color from a set of $t$ colors such that adjacent vertices must receive different colors. The *chromatic number* of $G$ denoted as $\chi(G)$ is the smallest $t$ of such a coloring admitted by $G$. Let $n \geqslant 2k$ be positive integers. The *Kneser graph* $\mathrm{KG}(n,k)$ has the vertex set $\binom{[n]}{k}$ of all $k$-subsets of $[n] = \{1, 2, 3, \ldots, n\}$, where two

*Corresponding author.
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vertices $A$ and $B$ are adjacent if $A \cap B = \emptyset$. Figure 1 shows an example of KG(5, 2) with a proper 3-coloring.

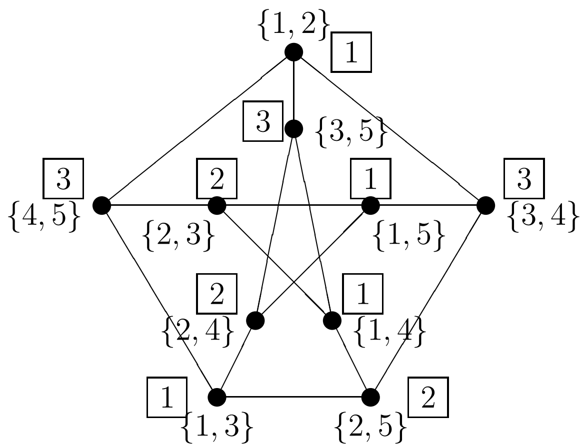


Figure 1: A proper 3-coloring of KG(5, 2) (also known as Petersen graph).

Lovász [15] in 1978 confirmed the Kneser conjecture [11] that the chromatic number of KG$(n, k)$ is equal to $n - 2k + 2$. Lovász's proof applied topological methods to a combinatorial problem. Since then, algebraic topology has became an important tool in combinatorics. In particular, various alternative proofs (cf. [2, 7, 17]) and generalizations (cf. [1, 12, 13, 16, 20, 21]) of the Lovász-Kneser theorem have been developed. Most of these proofs utilized methods or results in algebraic topology, mainly the Borsuk-Ulam theorem and its extensions.

**Theorem 1. (Lovász-Kneser Theorem [15])** *For any* $n \geqslant 2k$,

$$\chi(\mathrm{KG}(n, k)) = n - 2k + 2.$$

In 2004, Matoušek [17] gave a self-contained combinatorial proof for the Lovász-Kneser Theorem by utilizing the Tucker Lemma [23] together with a specialized constructive proof for the Tucker Lemma by Freund and Todd [6]. Later on, Ziegler [27] gave combinatorial proofs for various generalizations of the Lovász-Kneser Theorem.

For positive integers $p \geqslant 2q$, a $(p, q)$-*coloring* for a graph $G$ is a mapping $f : V(G) \to \{0, 1, 2, \ldots, p-1\}$ such that $|f(u) - f(v)|_p \geqslant q$ holds for adjacent vertices $u$ and $v$, where $|x|_p = \min\{|x|, p - |x|\}$. The *circular chromatic number* of $G$, denoted by $\chi_c(G)$, is the infimum $p/q$ of a $(p, q)$-coloring admitted by $G$. It is known (cf. [24, 25]) that $\chi_c(G)$ is rational if $G$ is finite, and the following hold for every graph $G$:

$$\chi(G) - 1 < \chi_c(G) \leqslant \chi(G). \tag{1.1}$$

Thus the circular chromatic number is a refinement of the chromatic number for a graph. The circular chromatic number reveals more information about the structure of a graph than the chromatic number does. Families of graphs for which the equality $\chi_c(G) = \chi(G)$ holds possess special structure properties and they have been broadly studied (cf. [24, 25]). Kneser graphs turned out to be an example among those widely studied families of graphs.

Johnson, Holroyd, and Stahl [10] conjectured that $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$. This conjecture has received much attention. The cases for $k = 2$, and $n = 2k+2$ was confirmed in [10]. By a combinatorial method, Hajiabolhassan and Zhu [9] proved that for a fixed $k$, the conjecture holds for sufficiently large $n$. Using topological approaches, Meunier [18] and Simonyi and Tardos [22] confirmed independently the case when $n$ is even. Indeed, all these results were proved true [9, 14, 18, 22] for the *Schrijver graph* $\mathrm{SG}(n,k)$, a subgraph of $\mathrm{KG}(n,k)$ induced by the $k$-subsets of $[n]$ that do not contain adjacent numbers modulo $n$. On the other hand, it was shown by Simonyi and Tardos [22] that for any $\epsilon > 0$, there exists $\delta > 0$ such that if $n$ is odd and $n - 2k \leqslant \delta k$, then $\chi_c(\mathrm{SG}(n,k)) \leqslant \chi(\mathrm{SG}(n,k)) - 1 + \epsilon$. Hence the Johnson-Holroyd-Stahl conjecture cannot be extended to Schrijver graphs.

In 2011, Chen [4] confirmed the Johnson-Holroyd-Stahl conjecture. A simplified proof for this result was given by Chang, Liu, and Zhu [3]. At the center of both proofs is the following:

**Lemma 2.** **(Alternative Kneser Coloring Lemma [4, 3]**) *Suppose* $c : \binom{[n]}{k} \to [n-2k+2]$ *is a proper coloring of* $\mathrm{KG}(n,k)$*. Then* $[n]$ *can be partitioned into three subsets,* $[n] = S \cup T \cup \{a_1, a_2, \ldots, a_{n-2k+2}\}$*, where* $|S| = |T| = k-1$*, and* $c(S \cup \{a_i\}) = c(T \cup \{a_i\}) = i$ *for* $i = 1, 2, \ldots, n-2k+2$*.*

Let $c$ be a proper $(n-2k+2)$-coloring of $\mathrm{KG}(n,k)$. The Lovász-Kneser Theorem is equivalent to saying that every color class in $c$ is non-empty. Lemma 2 strengthens this result by revealing the exquisite structure of a Kneser graph induced by an optimal coloring. For instance, the proper 3-coloring in Figure 1 has $a_i = i$ for $i = 1, 2, 3$, $S = \{4\}$, and $T = \{5\}$. By Lemma 2, the subgraph of $\mathrm{KG}(n,k)$ induced by the vertices $S \cup \{a_i\}$ and $T \cup \{a_i\}$, $1 \leqslant i \leqslant n-2k+2$, is a fully colored (i.e. uses all colors) complete bipartite graph $K_{n-2k+2,n-2k+2}$ minus a perfect matching. Moreover, the closed neighborhood for each vertex in this subgraph is fully colored.

It is known (cf. [8]) that this fact easily implies that $\chi_c(\mathrm{KG}(n,k)) = \chi(\mathrm{KG}(n,k))$. For completeness, we include a proof of this implication.

**Theorem 3.** [4, 3] *For positive integers* $n \geqslant 2k$*,* $\chi_c(\mathrm{KG}(n,k)) = n - 2k + 2$*.*

*Proof.* Assume to the contrary that $\chi_c(\mathrm{KG}(n,k)) = p/q$ where $\gcd(p,q) = 1$ and $q \geqslant 2$. Let $d = n - 2k + 2$. By (1.1), it must be $(d-1)q < p < dq$. Let $f$ be a $(p,q)$-coloring for $\mathrm{KG}(n,k)$. The function $c$ defined on $\binom{[n]}{k}$ by $c(v) = \lfloor f(v)/q \rfloor$ is a proper coloring of $\mathrm{KG}(n,k)$ using colors in $\{0, 1, 2, \ldots, d-1\}$.

By Lemma 2, there is a partition $[n] = S \cup T \cup \{a_0, a_1, \ldots, a_{n-2k+1}\}$ such that $c(S \cup \{a_i\}) = c(T \cup \{a_i\}) = i$ for $0 \leqslant i \leqslant n - 2k + 1$. Denote $S_i = S \cup \{a_i\}$ and $T_i = T \cup \{a_i\}$ for $i = 0, 1, \ldots, d - 1$. By the definition of $c$, we obtain

$$iq \leqslant f(S_i), f(T_i) < \min\{(i + 1)q, p\}, \text{ for } i = 0, 1, 2, \ldots, d - 1.$$

Assume $f(S_0) \geqslant f(T_0)$ (the other case can be proved similarly). Then $f(T_1) \geqslant f(S_0) + q$ and $f(S_2) \geqslant f(T_1) + q$, implying $f(S_2) \geqslant f(S_0) + 2q$. Continue this process until the last term. If $d$ is even, we obtain $f(T_{d-1}) \geqslant f(S_0) + (d - 1)q$. Because $S_0$ and $T_{d-1}$ are adjacent, so $|f(S_0) - f(T_{d-1})|_p \geqslant q$. This implies that $p - f(T_{d-1}) + f(S_0) \geqslant q$. Hence, $p \geqslant dq$, a contradiction.

If $d$ is odd, we obtain $f(S_{d-1}) \geqslant f(S_0) + (d - 1)q$. Because $T_0$ and $S_{d-1}$ are adjacent, so $|f(T_0) - f(S_{d-1})|_p \geqslant q$. This implies that $p - f(S_{d-1}) + f(T_0) \geqslant q$. Since $f(S_0) \geqslant f(T_0)$, so $p \geqslant dq$, a contradiction. Thus Theorem 3 follows. □

Both proofs of Lemma 2 in [4, 3] utilized Fan's lemma [5] applied to the boundary of the barycentric subdivision of $n$-cubes. The aim of this article is to present a proof for Lemma 2, which on one hand is a self-contained combinatorial proof, and on the other hand, further simplifies the proof presented in [3].

Our proof of Lemma 2, presented in the next two sections, is established by modifying a constructive proof for Fan's lemma given by Prescott and Su [19] to the desired special case, together with the labeling scheme used in [3]. The proof for the labeling scheme is further simplified and more straightforward than the one in [3]. In addition, our modification of the constructive proof in [19] corrects a minor error occurred in that paper.

## 2 Labeling of $\{0, 1, -1\}$-vectors

We present a proof of the Fan's lemma [5] applied to the boundary of the first barycentric subdivision of the $n$-cubes. The proof is by modifying and specializing the constructive proof of Fan's lemma given by Prescott and Su [19].

Let $n$ be a positive integer and $\mathcal{F}^n = \{0, 1, -1\}^n \setminus \{(0, 0, \ldots, 0)\}$ be the family of vectors $A = (a_1, a_2, \ldots, a_n)$, where each $a_i \in \{0, 1, -1\}$, and $a_j \neq 0$ for at least one $j$. A vector $A \in \mathcal{F}^n$ can also be expressed as $A = (A^+, A^-)$ where $A^+ = \{i : a_i = 1\}$ and $A^- = \{i : a_i = -1\}$. Let $|A| = |A^+| + |A^-|$. Notice that $A^+ \cap A^- = \emptyset$, and $|A| \geqslant 1$. For $A = (A^+, A^-), B = (B^+, B^-) \in \mathcal{F}^n$, we write $A \leqslant B$ if $A^+ \subseteq B^+$ and $A^- \subseteq B^-$. If $A \leqslant B$ but $A \neq B$, then $A < B$.

Let $n, m$ be positive integers. Let $\lambda$ be an $m$-labeling (mapping) from $\mathcal{F}^n$ to $\{\pm 1, \pm 2, \ldots, \pm m\}$. We say $\lambda$ is *anti-podal* if $\lambda(-X) = -\lambda(X)$ for all $X \in \mathcal{F}^n$. Two vectors $X, Y \in \mathcal{F}^n$ form a *complementary pair* if $X < Y$ and $\lambda(X) + \lambda(Y) = 0$. In the following, we assume that $\lambda$ is an anti-podal labeling of $\mathcal{F}^n$ without complementary pairs.

A non-empty subset $\sigma$ of $\mathcal{F}^n$ is called a *simplex* if the vectors in $\sigma$ can be ordered as $A_1 < A_2 < \cdots < A_d$. Since $|A_d| \leqslant n$, if $\sigma$ is a simplex, then $1 \leqslant |\sigma| \leqslant n$. Figure 2 shows an example of $\mathcal{F}^3$.

Topologically, each vector $A \in \mathcal{F}^n$ is a point on the boundary of the $n$-dimensional cube (with $a_i$ be the $i$th coordinate of the point), and a simplex $\sigma$ defined above is the convex hull of the points in $\sigma$. Although our proof does not use the topological meaning of this concept, this topological background can be helpful in understanding the arguments.

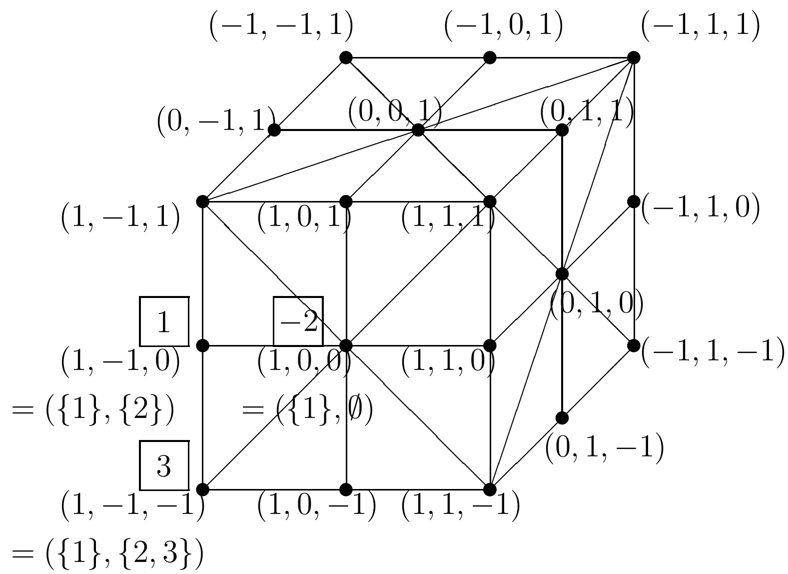


Figure 2: Vertices and points in $\mathcal{F}^3$, where each triangle is a simplex of three vertices. The boxed numbers (labels) show an example of a positive alternating simplex $\sigma : A_1 < A_2 < A_3$, where $A_1 = (1, 0, 0)$, $A_2 = (1, -1, 0)$, $A_3 = (1, -1, -1)$, and $\lambda(\sigma) = \{1, -2, 3\}$.

A simplex $\sigma = A_1 < A_2 < \cdots < A_d$ is *alternating* with respect to $\lambda$ if the set $\lambda(\sigma) = \{\lambda(A_1), \lambda(A_2), \ldots, \lambda(A_d)\}$ of labels can be expressed either as $\{k_1, -k_2, k_3, \ldots, (-1)^{d-1}k_d\}$ or as $\{-k_1, k_2, -k_3, \ldots, (-1)^d k_d\}$, where $1 \leqslant k_1 < k_2 < \cdots < k_d \leqslant m$. In the former case, $\text{sign}(\sigma) = 1$ and $\sigma$ is *positive alternating*; in the latter case, $\text{sign}(\sigma) = -1$ and $\sigma$ is *negative alternating*.

A simplex $\sigma$ is *almost-alternating* if it is not alternating, but the deletion of some element from $\sigma$ results in an alternating simplex. Since there are no complementary pairs, every almost-alternating simplex contains exactly two elements such that the deletion of each of them from $\sigma$ results in an alternating simplex. Moreover, both resulting alternating simplexes are of the same sign. This common sign is defined as $\text{sign}(\sigma)$.

The *maximum non-zero index* of a simplex, $\sigma = A_1 < \cdots < A_d$, is $\max(\sigma) = \max\{i :$ the $i$-th term of $A_d$ is non-zero$\}$. Denote $\beta(\sigma)$ as the $(\max(\sigma))$-th term of $A_d$. An alternating or almost-alternating simplex $\sigma$ is *agreeable* if $\beta(\sigma) = \text{sign}(\sigma)$.

**Lemma 4.** [5] *Assume $\lambda : \mathcal{F}^n \to \{\pm 1, \pm 2, \ldots, \pm m\}$ is an anti-podal labeling without complementary pairs. Then there exist an odd number of positive alternating simplexes of size $n$. Consequently, $m \geqslant n$.*

Figure 3 shows examples of Lemma 4 for $n = m = 2$.

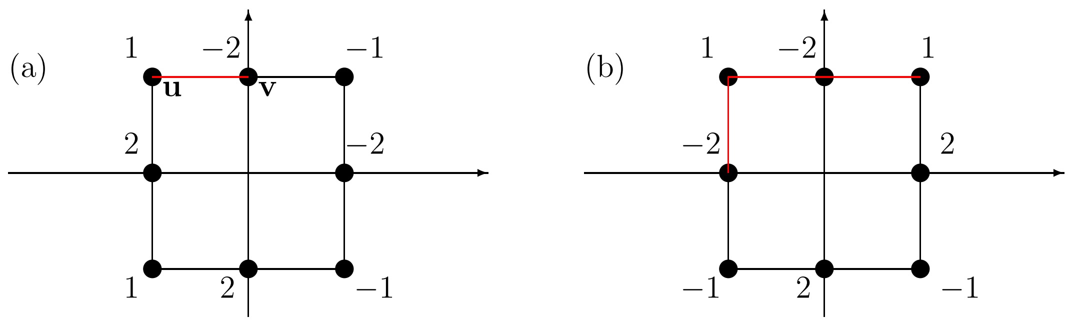


Figure 3: There are 8 vectors (points) in $\mathcal{F}^2$. In each (a) and (b), the numbers on the vectors form an anti-podal 2-labeling without complementary pairs. In (a) there is only one positive alternating simplex of size 2, namely $uv$, while in (b) there are three such simplexes.

*Proof.* Define a graph $G$ with the following three types of simplexes $\sigma$ as vertices.

Type I: $\max(\sigma) = |\sigma| + 1$, and $\sigma$ is agreeable alternating.

Type II: $\max(\sigma) = |\sigma|$, and $\sigma$ is agreeable almost-alternating.

Type III: $\max(\sigma) = |\sigma|$, and $\sigma$ is alternating.

Two vertices $\sigma$ and $\tau$ are adjacent in $G$ if all the following conditions are satisfied:

(1) $\sigma \subset \tau$, $|\sigma| = |\tau| - 1$,

(2) $\sigma$ is alternating,

(3) $\beta(\tau) = \text{sign}(\sigma)$, and

(4) $\max(\tau) = |\tau|$.

**Claim 1.** All vertices in $G$ have degree 2, except that Type III vertices with $|\sigma| = 1$ or $n$ have degree 1.

*Proof.* Let $\sigma$ be a Type I vertex with $\max(\sigma) = |\sigma| + 1 = d$. By Conditions (1) and (4), a neighbor $\tau$ of $\sigma$ must be a vertex of Type II or III and have $\max(\tau) = |\tau| = d$.

Since $|\sigma|+1=\max(\sigma)$, there exists a unique index $1\leqslant j\leqslant d$ such that the elements of $\sigma$ can be expressed as $A_1<\cdots<A_{j-1}<A_{j+1}<\cdots<A_d$, where $|A_i|=i$ for all $i$.

If $1\leqslant j<d$, then there exist two indices $1\leqslant t,r\leqslant d$ such that the $t$-th and the $r$-th terms are non-zero in $A_{j+1}$ (denoted by $a_t$ and $a_r$, respectively), but zero in $A_{j-1}$ (or $A_{j-1}$ does not exist in case $j=1$). Let $\tau_1=\sigma\cup A_j$ and $\tau_2=\sigma\cup A'_j$, where $A_j$ (or $A'_j$, respectively) is obtained by replacing the $t$-th (or $r$-th, respectively) term of $A_{j+1}$ by 0. Since $\sigma$ is agreeable alternating and there are no complementary pairs, each of $\tau_1$ and $\tau_2$ is a Type II or III vertex, and they are the only neighbors of $\sigma$ in $G$.

If $j=d$, then $\sigma=A_1<\cdots<A_{d-1}$, and $|A_i|=i$. Since $\max(\sigma)=d$, there exists a unique index $1\leqslant t<d$ such that the $t$-th term of all elements of $\sigma$ is 0. Hence, the only two neighbors of $\sigma$ are $\tau: A_1<\cdots<A_{d-1}<A_d$, where $A_d$ is either $(A^+_{d-1}\cup\{t\},A^-_{d-1})$ or $(A^+_{d-1},A^-_{d-1}\cup\{t\})$. Similar to the above discussion, each $\tau$ is a Type II or III vertex.

Let $\sigma$ be a Type II vertex. By (1) and (2), its neighbors $\tau$ must be alternating simplexes obtained from $\sigma$ by deleting one element. Since $\sigma$ is almost-alternating, there are exactly two elements such that the deletion of each from $\sigma$ results in an alternating simplex. Since $\sigma$ is agreeable, each of these two resulted alternating simplexes $\tau$ is either a vertex of Type I (if $\max(\tau)=\max(\sigma)$) or a vertex of Type III (if $\max(\tau)=\max(\sigma)-1$). Both are neighbors of $\sigma$.

Let $\sigma$ be a Type III vertex. By (1), a neighbor $\tau$ of $\sigma$ has $|\tau|=|\sigma|\pm 1$. Of course, if $|\sigma|=1$, then no neighbor $\tau$ of $\sigma$ has $|\tau|=|\sigma|-1$; if $|\sigma|=n$, then no neighbor $\tau$ of $\sigma$ has $|\tau|=|\sigma|+1$. Now we show that if $|\sigma|\geqslant 2$ (respectively, $|\sigma|\leqslant n-1$) then $\sigma$ has exactly one neighbor $\tau$ with $|\tau|=|\sigma|-1$ (respectively, with $|\tau|=|\sigma|+1$).

Assume $|\sigma|\geqslant 2$. If $\sigma$ is agreeable, then delete the element of $\sigma$ with the maximum absolute label in $\lambda(\sigma)$. If $\sigma$ is not agreeable, then delete the element with the minimum absolute label in $\lambda(\sigma)$. For each of the two cases, if the resulted simplex $\tau$ has $\max(\tau)=\max(\sigma)$, then $\tau$ is agreeable (since $\sigma$ is agreeable) so it is a vertex of Type I. If $\tau$ has $\max(\tau)=\max(\sigma)-1$, then $\tau$ is a vertex of Type III. In both cases, $\tau$ is a neighbor of $\sigma$. By (2) and (3), the deletion of any other element from $\sigma$ is not a neighbor of $\sigma$.

Now consider $|\sigma|\leqslant n-1$. Denote $\sigma=A_1<A_2<\ldots<A_d$, where $d\leqslant n-1$ and $A_d=(a_1,\ldots,a_d,0,\ldots,0)$. Let $A_{d+1}=(a_1,\ldots,a_d,\operatorname{sign}(\sigma),0,\ldots,0)$. Then $\tau=A_1<\cdots<A_d<A_{d+1}$ is a vertex of Type II or III, and is a neighbor of $\sigma$. By (3) and (4), $\tau$ is the only neighbor of $\sigma$ with an additional element.

In conclusion, each Type III vertex has degree 2 if $2\leqslant d\leqslant n-1$, and degree 1 if $d=1,n$. This completes the proof of Claim 1. □

By Claim 1, $G$ is a union of disjoint paths and cycles. The vertices of degree 1 are $\{(1,0,\ldots,0)\}$, $\{(-1,0,\ldots,0)\}$, and all alternating simplexes of size $n$. For each path $P=(\sigma_1,\sigma_2,\ldots,\sigma_t)$ in $G$, its *negation* $-P=(-\sigma_1,-\sigma_2,\ldots,-\sigma_t)$ is also a path in $G$. Here $-\sigma_i$ is the set obtained from $\sigma_i$ by negating each of its elements. Observe that $P\neq -P$, for otherwise, we must have $\sigma_t=-\sigma_1$, $\sigma_{t-1}=-\sigma_2$, and eventually we get either $\sigma_i=-\sigma_i$ or $\sigma_{i+1}=-\sigma_i$. Both are impossible. Hence the paths in $G$ come in pairs, resulting in an even number of paths in $G$. So $G$ has $4r$ vertices of degree 1, for some $r \geqslant 1$. Thus there are $4r-2$ alternating simplexes of size $n$. Observe that if $\sigma$ is a positive alternating simplex, then $-\sigma$ is a negative alternating simplex. Hence there are $2r-1$ positive alternating simplexes of size $n$. This completes the proof for Lemma 4. $\square$

Note that without Condition (4) in the above proof, Claim 1 does not hold. However, this condition was missing in the proof presented in [19], but was added in [26].

# 3 Proof of Lemma 2

We prove Lemma 2 by the same labeling used in [3]. However, the argument is further simplified. Let $c$ be a proper $(n-2k+2)$-coloring of $\mathrm{KG}(n,k)$ using colors from the set $\{2k-1, 2k, \ldots, n\}$. For a subset $A$ of $[n]$ with $|A| \geqslant k$, let

$$c(A) = \max\{c(U) \colon U \subseteq A, |U| = k\}.$$

Let $\prec$ be an arbitrary linear ordering of $2^{[n]}$ such that if $|X| < |Y|$, then $X \prec Y$. Let $\lambda$ be a labeling from $\mathcal{F}^n$ to $\{\pm 1, \pm 2, \ldots, \pm n\}$ defined by:

$$\lambda(A) = \begin{cases} |A|, & \text{if } |A| \leqslant 2k-2 \text{ and } A^- \prec A^+; \\ -|A|, & \text{if } |A| \leqslant 2k-2 \text{ and } A^+ \prec A^-; \\ c(A^+), & \text{if } |A| \geqslant 2k-1 \text{ and } A^- \prec A^+; \\ -c(A^-), & \text{if } |A| \geqslant 2k-1 \text{ and } A^+ \prec A^-. \end{cases}$$

Notice that if $|A| \geqslant 2k-1$, then $|A^+| \geqslant k$ or $|A^-| \geqslant k$. Hence, $\lambda$ is well-defined. Apparently, $\lambda$ is anti-podal. Suppose there exists a complementary pair $X < Y$ with $\lambda(X) = -\lambda(Y)$. That is, $X = (X^+, X^-)$ and $Y = (Y^+, Y^-)$, where $X^+ \subseteq Y^+$, $X^- \subseteq Y^-$, and it is not the case that $X^+ = Y^+$ and $X^- = Y^-$. As $X < Y$, so $|X| < |Y|$. Assume $\lambda(X) > 0$. (The other case is similar.) By definition of $\lambda$, it must be $|X|, |Y| \geqslant 2k-1$. Therefore, there exist $A, B \subseteq [n]$ such that $|A| = |B| = k$, $A \subseteq X^+ \subseteq Y^+$, $B \subseteq Y^-$, and $c(A) = c(B)$, which is impossible as $A \cap B = \emptyset$ (since $Y^+ \cap Y^- = \emptyset$). Thus there are no complementary pairs. By Lemma 4, there are an odd number of positive alternating simplexes of size $n$.

**Claim 2.** Assume $\sigma : X_1 < X_2 < \cdots < X_n$ is a positive alternating simplex with respect to $\lambda$. Then $|X^+_{2k-2}| = |X^-_{2k-2}| = k-1$, and $[n]$ can be partitioned as $[n] = X^+_{2k-2} \cup X^-_{2k-2} \cup \{a_{2k-1}, a_{2k}, \ldots, a_n\}$, where

$$\begin{aligned} &c(X^+_{2k-2} \cup \{a_{2k-1}, a_{2k+1}, \ldots, a_j\}) = j, && \text{if } j \text{ is odd;} \\ &c(X^-_{2k-2} \cup \{a_{2k}, a_{2k+2}, \ldots, a_j\}) = j, && \text{if } j \text{ is even.} \end{aligned}$$

*Proof.* By assumption, $\lambda(\sigma)=\{1,-2,\ldots,(-1)^{n-1}n\}$. So, $|X_i|=i$ for $1\leqslant i\leqslant n$. By definition of $\lambda$, $\lambda(X_i)=(-1)^{i-1}i$ for $1\leqslant i\leqslant 2k-2$, $|X^+_{2k-2}|=|X^-_{2k-2}|=k-1$, and $\lambda(\{X_{2k-1},\ldots,X_n\})=\{2k-1,-2k,\ldots,(-1)^{n-1}n\}$.

Let $q=\lceil\frac{n-2k+2}{2}\rceil$ and $q'=\lfloor\frac{n-2k+2}{2}\rfloor$. The set $\lambda(\{X_{2k-1},\ldots,X_n\})$ consists of $q$ positive labels and $q'$ negative labels. By the definition of $\lambda$, if $\lambda(X_i)$ is positive (respectively, negative), $X_i$ is obtained from $X_{i-1}$ by adding one element to $X^+_{i-1}$ (respectively, to $X^-_{i-1}$). Thus when $i$ changes from $2k-1$ to $n$, the sets $X^+_i$ (respectively, $X^-_i$) changed $q$ times (respectively, $q'$ times), each time a new element is added. Since the positive (respectively, negative) labels in $\lambda(\{X_{2k-1},\ldots,X_n\})$ are $\{2k-1,2k+1,\ldots,2(k+q-1)-1\}$ (respectively, $\{-2k,-(2k+2),\ldots,-(2(k+q'-1))\}$), by the monotonicity of $c$, each time when a new element is added to $X^+_i$ (or $X^-_i$, respectively), the value of $c(X^+_i)$ (or $c(X^-_i)$) increases by 2. Therefore $\{2k-1,2k,\ldots,n\}$ is partitioned into $I=\{j_1<j_2<\ldots<j_q\}$ and $I'=\{j'_1<j'_2<\ldots<j'_{q'}\}$ such that $\lambda(X_{j_t})=c(X^+_{j_t})=2k-2+2t-1$ and $\lambda(X_{j'_t})=-c(X^-_{j'_t})=-(2k-2+2t)$. Moreover $X^+_{j_t}$ is obtained from $X^+_{j_{t-1}}$ by adding one element, and $X^-_{j_t}$ is obtained from $X^-_{j_{t-1}}$ by adding one element. So Claim 2 follows. □

Let $\Gamma$ be the family of vectors $X$ with $|X^+|=|X^-|=k-1$. By Claim 2, each positive alternating simplex of size $n$ contains exactly one element in $\Gamma$. For $W\in\Gamma$, let $\alpha(W,\lambda)$ be the number of positive alternating simplexes of size $n$ with respect to $\lambda$, containing $W$ as an element. By Lemma 4, $\Sigma_{X\in\Gamma}\alpha(X,\lambda)$ is odd. Hence there exists $Z\in\Gamma$ such that $\alpha(Z,\lambda)$ is odd. Let $\sigma: X_1<X_2<\cdots<X_n$ be a positive alternating simplex with respect to $\lambda$, where $Z=X_{2k-2}$. Let $Z=(Z^+,Z^-)=(S,T)$.

Define $\lambda':\mathcal{F}^n\to\{\pm1,\pm2,\ldots,\pm n\}$ by

$$\lambda'(X)=\begin{cases}-\lambda(X), & \text{if } X\in\{Z,-Z\};\\ \lambda(X), & \text{otherwise.}\end{cases}$$

Similar to $\lambda$, $\lambda'$ is also anti-podal without complementary pairs. Moreover, Claim 2 holds for $\lambda'$. By Lemma 4, $\Sigma_{X\in\Gamma}\alpha(X,\lambda')$ is odd. Since $\alpha(X,\lambda')=\alpha(X,\lambda)$ for $X\in\Gamma\setminus\{Z,-Z\}$, so $\alpha(Z,\lambda)+\alpha(-Z,\lambda)\equiv\alpha(Z,\lambda')+\alpha(-Z,\lambda')\pmod 2$. Because $\lambda(-Z)=2k-2=\lambda'(Z)$, we get $\alpha(-Z,\lambda)=\alpha(Z,\lambda')=0$, implying $\alpha(-Z,\lambda')\equiv\alpha(Z,\lambda)\equiv 1\pmod 2$. Hence, there exists a positive alternating simplex $\tau: Y_1<\cdots<Y_n$ with respect to $\lambda'$, where $Y_{2k-2}=-Z=(T,S)$. Apply Claim 2 to $\sigma$ and $\tau$, we obtain for $2k-1\leqslant i\leqslant n$:

$$\begin{aligned}&c(S\cup\{a_{2k-1},a_{2k+1},\ldots,a_i\})=c(T\cup\{b_{2k-1},b_{2k+1},\ldots,b_i\})=i, && \text{for odd } i;\\ &c(T\cup\{a_{2k},a_{2k+2},\ldots,a_i\})=c(S\cup\{b_{2k},b_{2k+2},\ldots,b_i\})=i, && \text{for even } i,\end{aligned}$$

where $\{a_{2k-1},a_{2k},\ldots,a_n\}=\{b_{2k-1},b_{2k},\ldots,b_n\}=[n]\setminus(S\cup T)$.

To complete the proof for Lemma 2, it remains to show: For any index $2k-1\leqslant i\leqslant n$, it holds that $a_i=b_i$ and $c(S\cup\{a_i\})=c(T\cup\{a_i\})=i$. We verify this by induction on $i$. Assume $i = 2k-1$. As $c(S \cup \{a_{2k-1}\}) = c(T \cup \{b_{2k-1}\}) = 2k-1$, so $S \cup \{a_{2k-1}\}$ and $T \cup \{b_{2k-1}\}$ are not adjacent, implying $a_{2k-1} = b_{2k-1}$. Similarly, it holds for $i = 2k$.

Assume $i \geqslant 2k+1$ and the result holds for $j < i$. If $i$ is odd, as $S \cup \{a_i\}$ is adjacent to $T \cup \{a_j\}$ for all $2k-1 \leqslant j < i$, it follows that $c(S \cup \{a_i\}) \neq c(T \cup \{a_j\}) = j$ for $2k-1 \leqslant j < i$. Thus, $c(S \cup \{a_i\}) = i$, as $c(S \cup \{a_i\}) \leqslant i$. Similarly, we get $c(T \cup \{b_i\}) = i$. Hence, $S \cup \{a_i\}$ and $T \cup \{b_i\}$ are not adjacent, implying $a_i = b_i$. The case for even $i$ is obtained similarly. This completes the proof for Lemma 2. □

Note that according to (1.1), Theorem 3 implies the Lovász-Kneser Theorem. Moreover, Lovász-Kneser Theorem can be derived directly from Lemma 4. Assume to the contrary, $\chi(\mathrm{KG}(n,k)) \leqslant n-2k+1$. Let $c$ be a proper coloring for $\mathrm{KG}(n,k)$ using colors from $\{2k-1, 2k, \ldots, n-1\}$. Let $\lambda$ be the same labeling defined in our proof, except in this case $\lambda$ is from $\mathcal{F}^n$ to $\{\pm 1, \pm 2, \ldots, \pm(n-1)\}$, instead of to $\{\pm 1, \pm 2, \ldots, \pm n\}$. By the same argument, $\lambda$ is anti-podal without complementary pairs, contradicting Lemma 4 (as $n-1 < n$).

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