# Ch 1: INTRODUCTION TO GROUPS

Thurs. Sep 26 '13

## Section 1.1: Basic Axioms & Examples

**Def:** A **GROUP** is a set $G$ w/ a binary operation $*$ such that the following are satisfied:

1.) **CLOSURE:** If $a, b \in G$, then $a * b \in G$
2.) **ASSOCIATIVITY:** $(a*b)*c = a*(b*c)$ $\forall a,b,c \in G$
3.) **IDENTITY:** $\exists e \in G$ s.t. $a*e = e*a = a$ $\forall a \in G$.
4.) **INVERSES:** $\forall a \in G$, $\exists a^{-1} \in G$ s.t. $a*a^{-1} = a^{-1}*a = e$.

**Def:** A group $G$ is **ABELIAN** if $a*b \overset{\text{COMMUTATIVE}}{=} b*a$ $\forall a,b \in G$.

**Notation:** We write $a*b$ as $ab$ when dealing w/ an abstract group.

Ex.) $\mathbb{Z} = \{\ldots, -2, -1, 0, 1, 2, \ldots\}$ under $+$.

1.) If $a, b \in \mathbb{Z}$, then $a + b \in \mathbb{Z}$ (CLOSED UNDER +)
2.) $(a+b)+c = a+(b+c)$ $\forall a,b,c \in \mathbb{Z}$ (ASSOCIATIVE)
3.) $e = 0 \in \mathbb{Z}$, since $a + 0 = 0 + a = a$ $\forall a \in \mathbb{Z}$ (IDENTITY)
4.) $a^{-1} = -a \in \mathbb{Z}$, since $a + (-a) = (-a) + a = 0$ $\forall a \in \mathbb{Z}$ (ADDITIVE INVERSES)
5.) $a + b = b + a$ $\forall a, b \in \mathbb{Z}$ (COMMUTATIVE)

$\therefore \langle \mathbb{Z}, + \rangle$ is an Abelian Group.

Ex.)

| GROUPS | NON-GROUPS |
|---|---|
| $\langle \mathbb{R}, + \rangle$ | $\langle \mathbb{R}, \cdot \rangle$ $\because$ 0 is NOT invertible! |
| $\langle \mathbb{R} \setminus \{0\}, \cdot \rangle$ | $\langle \mathbb{Z}, \cdot \rangle$ |
| $\langle \mathbb{C}, + \rangle$ | $\langle \mathbb{C}, \cdot \rangle$ |
| $\langle \mathbb{Q}, + \rangle$ | $\langle \mathbb{Q}, \cdot \rangle$ |

**Prop:** Let $G$ be a group. Then...

1.) $\exists!\, e \in G$ s.t. $e*a = a*e = a$ $\forall a \in G$ (UNIQUE IDENTITY)

Pf: Let $e, e' \in G$ be identity elements.
$\Rightarrow e = ee' = e'$ $\square$

2.) $\forall a \in G$, $\exists!\, a^{-1} \in G$ s.t. $a*a^{-1} = a^{-1}*a = e$ (UNIQUE INVERSES)

Pf: Let $a \in G$, and let $b, c \in G$ be inverses of $a$.
$\Rightarrow b = be = b(ac) = (ba)c = ec = c$ $\square$