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MATH 540A
Due: 11/21/07

4.5 - 5, 30 5.2 - A

4.5

5) [Show that a Sylow p-subgroup of D_{2n} is cyclic and normal for every odd prime p. Hint: Show that $\langle r^{\frac{n}{p^\alpha}} \rangle$ is the unique Sylow p-group of D_{2n} if p is odd.]

Pf: Let p be an odd prime. Let $\alpha \geq 0$ and $m \geq 1$ be integers such that $|D_{2n}| = 2n = p^\alpha m$. Since p is an odd prime, p^α must divide n (this holds for all $\alpha \geq 0$). So, there is a $k \in \mathbb{Z}^+$ such that $p^\alpha k = n$.

Now consider the cyclic subgroup $\langle r^k \rangle$:

$$|\langle r^k \rangle| \underset{\uparrow}{=} \frac{n}{\gcd(n,k)} = \frac{n}{k} = p^\alpha$$

(since $\mathbb{Z}_n \cong \{1, r, \ldots, r^{n-1}\} \leq D_{2n}$)

So, $\langle r^k \rangle$ is a Sylow p-subgroup of D_{2n}. Thus, $n_p \geq 1$. Now let's show that $\langle r^k \rangle$ is normal. Let $r^{ka} \in \langle r^k \rangle$, $a \in \mathbb{Z}$. Let $b \in \mathbb{Z}$. Then,

$$r^b r^{ka} r^{-b} = r^{b+ka-b} = r^{ka} \in \langle r^k \rangle, \text{ and}$$
$$sr^b r^{ka} (sr^b)^{-1} = sr^b r^{ka} r^{-b} s = sr^{ka} s = ssr^{-ka} = r^{-ka} \in \langle r^k \rangle.$$

Thus, $g \langle r^k \rangle g^{-1} \leq \langle r^k \rangle$ for all $g \in D_{2n}$. So, $\langle r^k \rangle$ is normal in D_{2n}. Thus, $n_p = 1$, so this is the unique Sylow p-subgroup of D_{2n}. ▨

DB2

(45 cont)

30) [How many elements of order 7 must there be in a simple group of order 168?]

Let G be a simple group of order $168 = 2^3 \cdot 3 \cdot 7$. Then by Sylow's Thm, we must have at least one Sylow 7-subgroup, and $n_7 \equiv 1 \pmod 7$. Also, $n_7 \mid 2^3 3 = 24$. So, $n_7 = 1$ or 8. If $n_7 = 1$, we would have a normal subgroup of order 7, contradicting our assumption that G is simple. So, $n_7 = 8$. Each of these subgroups have prime order and thus are cyclic. Let $P \in Syl_7(G)$. Then P is generated by every non-identity element of P, of which there are 6. Note that if $Q \in Syl_7(G)$, and $x \in P$, $x \in Q$, and $x \neq 1$, then $P = \langle x \rangle = Q$. This says that Sylow 7-subgroups don't share any non-identity elements in common, since if they did, they would be the same subgroup.

We have 8 Sylow 7-subgroups, each of which have 6 generators (elements of order 7). So, G has $8 \times 6 = \boxed{48}$ elements of order 7.

DB3

[5.2] A) [Let G be a finite abelian group. Prove that G is simple iff G is isomorphic to $\mathbb{Z}_p$ for some prime p.]

Pf: ($\Rightarrow$) Suppose G is simple. Then the only normal subgroups of G are $\{1\}$ and G. But since G is abelian, all subgroups are normal. So $\{1\}$ and G are the only subgroups. Since $|G| \geq 2$ (if $\{1\} \neq G$) we can pick $g \in G$ such that $g \neq 1$. Since $\langle g \rangle = G$, G is cyclic. So, $G \cong \mathbb{Z}_p$ for some $p \in \mathbb{Z}^+$. Now, p must be prime since if it weren't we would have $p = ab$ for $a, b \in \mathbb{Z}^+$, which would mean that we have subgroups of order a and b, but we don't! So, $G \cong \mathbb{Z}_p$ for some prime p.

namely, $\langle g^{\frac{p}{a}} \rangle$ and $\langle g^{\frac{p}{b}} \rangle$

($\Leftarrow$) Suppose $G \cong \mathbb{Z}_p$ for some prime p. Then since $|G| = p$, Lagrange tells us that G can only have subgroups of order 1 and p. These subgroups are $\{1\}$ and G. They are normal since G is abelian. So, G is simple.

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