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MATH 540A
Due: 10/17/07

3.1 - 5, 20   3.2 - 5

$\boxed{3.1}$

5) [Use the preceding exercise to prove that the order of the element gN in G/N is n, where n is the smallest positive integer such that $g^n \in N$ (and gN has infinite order if no such positive integer exists). Give an example to show that the order of gN in G/N may be strictly smaller than the order of g in G.]

Pf: The preceding exercise stated that in the quotient group, G/N, $(gN)^\alpha = g^\alpha N$ for all $\alpha \in \mathbb{Z}$. (*)

Now, suppose n is the smallest positive integer such that $g^n \in N$. By (*), we get that $(gN)^n = g^n N$. But since $g^n \in N$, $g^n N = N$. So, $|gN| \leq n$. Suppose $|gN| = m < n$. Then $(gN)^m = N$. But by (*) we have that $(gN)^m = g^m N$. So, since $g^m N = N$, $g^m \in N$. But this contradicts our assumption about n, so we must have that $|gN| = n$.

In the case that there is no positive integer n such that $g^n \in N$. Then, for all n, $g^n \notin N$. So, for all n, $(gN)^n = g^n N \neq N$. Thus gN has infinite order. ▨

Consider $G = \mathbb{Z}_4$ and $N = \{\bar{0}, \bar{2}\}$. Then $G/N = \{N, \bar{1}+N\}$. Here, $|\bar{1}+N| = 2$ but $|\bar{1}| = 4$, so $|\bar{1}+N| < |\bar{1}|$.

(3.1 cont) 20) [Let $G = \mathbb{Z}/24\mathbb{Z}$ and let $\tilde{G} = G/\langle\overline{12}\rangle$, where for each integer a we simplify notation by writing $\tilde{\bar{a}}$ as $\tilde{a}$.]

a) [Show that $\tilde{G} = \{\tilde{0}, \tilde{1}, \ldots, \widetilde{11}\}$.]

Since $G = \{\bar{0}, \bar{1}, \ldots, \overline{23}\}$, we have that

(Note: $\overline{12} + \langle\overline{12}\rangle = \langle\overline{12}\rangle$)

$$\tilde{G} = \{\langle\overline{12}\rangle, \bar{1}+\langle\overline{12}\rangle, \ldots, \overline{11}+\langle\overline{12}\rangle\}$$

$$= \{\{\bar{0},\overline{12}\}, \{\bar{1},\overline{13}\}, \ldots, \{\overline{11},\overline{23}\}\}$$

Here, pick a representative from each equivalence class.

$$= \{\tilde{\bar{0}}, \tilde{\bar{1}}, \ldots, \tilde{\overline{11}}\}$$

$$= \{\tilde{0}, \tilde{1}, \ldots, \widetilde{11}\}$$

Since $\tilde{\bar{a}} = \tilde{a}$

b) [Find the order of each element of $\tilde{G}$.]

$\tilde{0} + \tilde{0} = \langle\overline{12}\rangle + \langle\overline{12}\rangle = \langle\overline{12}\rangle = \tilde{0}$

$\tilde{0}$ must be the identity, so $\boxed{|\tilde{0}| = 1}$.

(By #5(*), since G and $\tilde{G}$ are abelian and therefore normal)

In general, $\tilde{a} = \bar{a} + \langle\overline{12}\rangle$, so $(\tilde{a})^n = (\bar{a} + \langle\overline{12}\rangle)^n = n\bar{a} + \langle\overline{12}\rangle$.
So only when $n\bar{a} = \bar{0}$ or $n\bar{a} = \overline{12}$ do we get that $(\tilde{a})^n = \langle\overline{12}\rangle$.
So,

$|\tilde{1}| = 12$ (since $n\bar{1} = \overline{12}$ when $n = 12$)

$|\tilde{2}| = 6$ (since $n\bar{2} = \overline{12}$ when $n = 6$)

$|\tilde{3}| = 4$

$|\tilde{4}| = 3$

$|\tilde{5}| = 12$ (since $n\bar{5} = \overline{60} = \bar{0}$ when $n = 12$)

$|\tilde{6}| = 2$

$|\tilde{7}| = 12$

$|\tilde{8}| = 3$ (since $3 \cdot \bar{8} = \overline{24} = \bar{0}$)

$|\tilde{9}| = 4$

$|\widetilde{10}| = 6$

$|\widetilde{11}| = 12$

DB3

(3.1 cont) (20 cont)

c) [Prove that $\tilde{G} \cong \mathbb{Z}/12\mathbb{Z}$ (Thus $(\mathbb{Z}/24\mathbb{Z})/(12\mathbb{Z}/24\mathbb{Z}) \cong \mathbb{Z}/12\mathbb{Z}$, just as if we inverted and cancelled the $24\mathbb{Z}$'s.]

Pf: Let $\phi : \tilde{G} \to \mathbb{Z}/12\mathbb{Z}$ be defined as $\phi(\tilde{x}) = \bar{x}$. Then for $\tilde{x}, \tilde{y} \in \tilde{G}$,

$$\phi(\tilde{x}+\tilde{y}) = \phi(\widetilde{x+y})$$ (since $\tilde{x}$ and $\tilde{y}$ are equivalence classes)

$$= \overline{x+y}$$

$$= \bar{x}+\bar{y}$$

$$= \phi(\tilde{x}) + \phi(\tilde{y})$$

Thus, ϕ is a homomorphism.

Now let $\psi : \mathbb{Z}/12\mathbb{Z} \to G$ be defined as $\psi(\bar{x}) = \tilde{x}$. Then

$\phi(\psi(\bar{x})) = \phi(\tilde{x}) = \bar{x}$, and
$\psi(\phi(\tilde{x})) = \psi(\bar{x}) = \tilde{x}$, for all $\bar{x} \in G$ and $\tilde{x} \in \tilde{G}$.

So, $\psi = \phi^{-1}$, so ϕ is bijective. Thus, ϕ is an isomorphism, so $\tilde{G} \cong \mathbb{Z}/12\mathbb{Z}$. ∎

DB4

3.2

5) [Let H be a subgroup of G and fix some element $g \in G$.]

a) [Prove that gHg^{-1} is a subgroup of G of the same order as H.]

Pf: Since $1 \in H$, $g1g^{-1} = gg^{-1} = 1 \in gHg^{-1}$, so $gHg^{-1} \neq \emptyset$. Now let gxg^{-1} and gyg^{-1} be in gHg^{-1}. Then $(gyg^{-1})^{-1} = gy^{-1}g^{-1} \in gHg^{-1}$ since

$$(gyg^{-1})(gy^{-1}g^{-1}) = gy1y^{-1}g^{-1} = gyy^{-1}g^{-1} = g1g^{-1} = gg^{-1} = 1, \text{ and}$$
$$(gy^{-1}g^{-1})(gyg^{-1}) = gy^{-1}1yg^{-1} = gy^{-1}yg^{-1} = g1g^{-1} = gg^{-1} = 1.$$

Thus,

$$\begin{aligned}(gxg^{-1})(gyg^{-1})^{-1} &= (gxg^{-1})(gy^{-1}g^{-1}) \\ &= gx1y^{-1}g^{-1} \\ &= gxy^{-1}g^{-1} \in gHg^{-1} \quad (\text{since } xy^{-1} \in H).\end{aligned}$$

So, $gHg^{-1} \leq G$.

To show that $|H| = |gHg^{-1}|$, we can create a bijection. Let $\phi: H \to gHg^{-1}$ be defined by $\phi(h) = ghg^{-1}$ for $h \in H$. Let $\psi: gHg^{-1} \to H$. Then

$$\phi(\psi(ghg^{-1})) = \phi(h) = ghg^{-1}, \text{ and}$$
$$\psi(\phi(h)) = \psi(ghg^{-1}) = h.$$

Thus, $\psi = \phi^{-1}$, so ϕ is a bijection. Consequently,

$|H| = |gHg^{-1}|$. ▨

(3.2 cont) (5 cont)

b) [Deduce that if $n \in \mathbb{Z}^+$ and H is the unique subgroup of G of order n then $H \trianglelefteq G$.]

Pf: Proceed by contraposition. Assume that H is not a normal subgroup of G. Then there is a $g \in G$ such that $H \neq gHg^{-1}$. But by part (a), $|H| = |gHg^{-1}| = n$. Thus, there are at least two subgroups of order n. $\blacksquare$