**Joint Distributions**

**Def:** Let $X$ and $Y$ be two discrete random variables defined on the same sample space. The function

$$p(x, y) = P(X = x, Y = y)$$

means: probability $X = x$ and $Y = y$ is called the joint probability function of $X$ and $Y$.

Let $A$ be the possible values of $X$ and $B$ be the possible values of $Y$.

**Ex.** (Using a table)

In a certain suburban area, each household reported the number of cars and TV sets owned. Let $X$ be the number of cars owned by a randomly selected household, let $Y$ be the number of TV sets owned by a randomly selected household.

Suppose we have the following data which makes our joint p.f.

<table>
<thead>
<tr>
<th>$X$</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>0.1</td>
<td>0</td>
<td>0.1</td>
<td>0</td>
</tr>
<tr>
<td>2</td>
<td>0.3</td>
<td>0</td>
<td>0.1</td>
<td>0.2</td>
</tr>
<tr>
<td>3</td>
<td>0</td>
<td>0.2</td>
<td>0</td>
<td>0</td>
</tr>
</tbody>
</table>

The functions $p_x(x) = \sum_{y \in B} p(x, y)$ and $p_y(y) = \sum_{x \in A} p(x, y)$ are called, respectively, the marginal probability functions of $X$ and $Y$. 

[Table and graph are hand-drawn with various values and functions indicated.]
So for example
\[ p(Z=1) = P(X=2, Y=1) = 0.3 \]

What's the probability that a randomly chosen household owns exactly 1 can? 
\[ P(X=1) = \sum_{y=1}^{4} f(1, y) = f(1, 1) + f(1, 2) + f(1, 3) + f(1, 4) \]
\[ = 0.1 + 0 + 0.1 + 0 = 0.2 \]

**Ex.** Roll a 6-sided die and let the outcome be \( X \). Then toss a fair coin \( X \) times and let \( Y \) denote the number of tails, and let \( Z \) denote the number of heads. Find the joint p.f. of \( X \) and \( Y \) and the marginal probability functions of \( X \) and \( Y \).

**Solution:** Note that \( X \) can lie in \( A = \{1, 2, 3, 4, 5, 6\} \) and \( Y \) can lie in \( B = \{0, 1, 2, 3, 4, 5, 6\} \).
Case 1: \( \overline{X} = 1 \)

If \( \overline{X} = 1 \), then \( X = 0 \) or \( X = 1 \).

And

\[
P(1,0) = P(X=1, \overline{X}=0) = P(X=1) \cdot P(\overline{X}=0 | X=1)
\]

\[
= \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}
\]

\[
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\]

\[
= \frac{1}{6} \cdot \frac{1}{2} = \frac{1}{12}.
\]

Case 2: \( \overline{X} = 2 \)

If \( \overline{X} = 2 \), then \( X = 0 \), \( X = 1 \), or \( X = 2 \).

\[
P(2,0) = P(X=2, \overline{X}=0) = P(X=2)P(\overline{X}=0 | X=2)
\]

\[
= \frac{1}{6} \cdot \left( \frac{3}{4} \right) \cdot \left( \frac{1}{2} \right)^0 \left( \frac{1}{2} \right)^3 = \frac{1}{48}
\]

Similarly \( P(2,1) = \frac{1}{12} \) and \( P(2,2) = \frac{1}{24} \).

Case 3: \( \overline{X} = 3 \)

If \( \overline{X} = 3 \), then \( X = 0 \), \( X = 1 \), \( X = 2 \), or \( X = 3 \).

\[
P(3,0) = P(X=3, \overline{X}=0) = P(X=3)P(\overline{X}=0 | X=3)
\]

\[
= \frac{1}{6} \cdot \left( \frac{3}{4} \right) \cdot \left( \frac{1}{2} \right)^0 \left( \frac{1}{2} \right)^3 = \frac{1}{48}
\]

\[
P(3,1) = P(X=3, X=1) = P(X=3)P(X=1 | X=3)
\]

\[
= \frac{1}{6} \cdot \left( \frac{3}{4} \right) \cdot \left( \frac{1}{2} \right)^1 = \frac{3}{48}
\]
Similar calculations yield the following table for $p(x,y)$.

<table>
<thead>
<tr>
<th></th>
<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
<th>6</th>
<th>$p_x(x)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$1/2$</td>
<td>$1/2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$1/6$</td>
</tr>
<tr>
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<td>$2/2$</td>
<td>$1/2$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$1/6$</td>
</tr>
<tr>
<td>3</td>
<td>$1/4$</td>
<td>$3/4$</td>
<td>$3/4$</td>
<td>$3/4$</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>$1/6$</td>
</tr>
<tr>
<td>4</td>
<td>$1/9$</td>
<td>$4/9$</td>
<td>$6/9$</td>
<td>$4/9$</td>
<td>$4/9$</td>
<td>0</td>
<td>0</td>
<td>$1/6$</td>
</tr>
<tr>
<td>5</td>
<td>$1/92$</td>
<td>$5/92$</td>
<td>$10/92$</td>
<td>$10/92$</td>
<td>$5/92$</td>
<td>$1/92$</td>
<td>0</td>
<td>$1/6$</td>
</tr>
<tr>
<td>6</td>
<td>$1/384$</td>
<td>$6/384$</td>
<td>$15/384$</td>
<td>$20/384$</td>
<td>$15/384$</td>
<td>$6/384$</td>
<td>$1/384$</td>
<td>$1/6$</td>
</tr>
</tbody>
</table>

$p_x(x)$ and $p_y(y)$ are gotten by summing the rows and columns of this table, respectively,
Def: Two random variables $X$ and $Y$, defined on the same sample space, have a continuous joint distribution if there exists a non-negative function $f(x,y)$ on the $xy$-plane, such that for any region $R$ of the $xy$-plane that can be formed from rectangles by a countable number of set operations,

$$ P((X,Y) \in R) = \iiint_{R} f(x,y) \, dx \, dy $$

The function $f(x,y)$ is called the joint probability density function of $X$ and $Y$, if this is the case then the functions

$$ f_{X}(x) = \int_{-\infty}^{\infty} f(x,y) \, dy $$

and

$$ f_{Y}(y) = \int_{-\infty}^{\infty} f(x,y) \, dx $$

are called the marginal density functions of $X$ and $Y$. 
Note: The marginal p.d.f.'s are explained as follows.

Let $B$ be a subset of $\mathbb{R}$.

Then

$$P(Y \in B) = P(-\infty \leq X \leq \infty, Y \in B)$$

$$= \int_B \left( \int_{-\infty}^{\infty} f(x,y) \, dx \right) \, dy$$

$f_{X}(y) \rightarrow$ distribution function for $Y$
Ex: The joint p.d.f. of \( X \) and \( Y \) is given by
\[
f(x,y) = \begin{cases} 
\lambda x y^2 & 0 \leq x \leq y \leq 1 \\
0 & \text{otherwise}
\end{cases}
\]

(a) Determine the value of \( \lambda \).
(b) Find the marginal probability density functions of \( X \) and \( Y \).

(a) To find \( \lambda \) we use \( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} f(x,y) \, dx \, dy = 1 \).

So,
\[
1 = \int_{0}^{1} \int_{x}^{1} \lambda x y^2 \, dy \, dx = \lambda \int_{0}^{1} \left[ \frac{1}{3} y^3 \right]_{x}^{1} x \, dx
\]
\[
= \lambda \int_{0}^{1} \left( \frac{1}{3} - \frac{1}{3} x^3 \right) x \, dx = \frac{\lambda}{10}
\]

So, \( \lambda = 10 \).

(b) \( f_X(x) = \int_{-\infty}^{\infty} f(x,y) \, dy = \int_{x}^{1} 10x y^2 \, dy \)
\[
= \frac{10}{3} x (1-x^3) \quad 0 \leq x \leq 1
\]
\[f_Y(y) = \int_{-\infty}^{\infty} f(x,y) \, dx = \int_{0}^{y} 10x y^2 \, dx = 5y^4, \quad 0 \leq y \leq 1\]