HW 4 Solutions

1) Let \( X \) = amount of money won or lost

(a) 
\[
P(X = -1) = \frac{5 \cdot 5 \cdot 5}{6^3} = \frac{125}{216}
\]
\[
\text{no die matches your number}
\]

\[
P(X = 1) = \frac{1 \cdot 5 \cdot 5 + 5 \cdot 1 \cdot 5 + 5 \cdot 5 \cdot 1}{6^3} = \frac{75}{216}
\]
\[
\text{exactly one die matches}
\]

\[
P(X = 2) = \frac{1 \cdot 1 \cdot 5 + 1 \cdot 5 \cdot 1 + 5 \cdot 1 \cdot 1}{6^3} = \frac{15}{216}
\]
\[
\text{exactly two dice}
\]

\[
P(X = 3) = \frac{1 \cdot 1 \cdot 1}{6^3} = \frac{1}{216}
\]
\[
\text{all three dice match your number}
\]

(b) \( P(\xi) = P(X = \xi) \)
(c) \[ E[\bar{X}] = (-1) \left( \frac{125}{216} \right) + (1) \left( \frac{75}{216} \right) + (2) \left( \frac{15}{216} \right) + (3) \left( \frac{1}{216} \right) \]

\[ = -\frac{17}{216} \approx -0.0787 \]

You should expect to lose 7.9 cents per play on average.

\[ S = \{ (3), (1,3), (2,3), (4,3), (1,1,3), (1,2,3), (1,4,3), (2,1,3), (2,2,3), (2,4,3), \ldots \} \]

\[ P(a_1, a_2, \ldots, a_n) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \left( \frac{1}{4} \right)^n \]

For example, \[ P(1,2,3) = \frac{1}{4} \cdot \frac{1}{4} \cdot \frac{1}{4} = \frac{1}{64} \]
Let's verify that $P$ is a probability function. That is, we must check that $\sum_{w \in S} P(w) = 1$.

$$\sum_{w \in S} P(w) = P(3) + P((1,3)) + P((2,3)) + P((4,3)) +$$

$$\{ P((1,1,3)) + P((1,2,3)) + P((1,3,3)) + P((2,1,3)) +$$

$$P((2,2,3)) + P((2,3,3)) + P((4,1,3)) + P((4,2,3)) +$$

$$\cdots$$

$$= \frac{1}{4} + 3 \cdot \left(\frac{1}{4}\right)^2 + 3^2 \left(\frac{1}{4}\right)^3 + 3^3 \left(\frac{1}{4}\right)^4 + 3^4 \left(\frac{1}{4}\right)^5 + \cdots$$

$$= \frac{1}{4} \left[ 1 + \left(\frac{3}{4}\right) + \left(\frac{3}{4}\right)^2 + \left(\frac{3}{4}\right)^3 + \cdots \right]$$

$$= \frac{1}{4} \left[ \frac{1}{1 - 3/4} \right] = 1.$$

(b) $A = \{(1,1,3), (1,2,3), (1,3,3), (2,1,3), (2,2,3), (2,3,3), (4,1,3), (4,2,3), (4,3,3)\}$

$P(A) = 9 \cdot \left(\frac{1}{4}\right)^3 = \frac{9}{64} \approx 0.1406 \approx 14\%$

(c) $B = \{(3), (1,3), (2,3), (4,3), (1,1,3), (1,2,3), (1,3,3), (2,1,3), (2,2,3), (2,3,3), (4,1,3), (4,2,3), (4,3,3)\}$

$P(B) = \frac{1}{4} + 3 \left(\frac{1}{4}\right)^2 + 9 \left(\frac{1}{4}\right)^3 = \frac{4^2 + 3 \cdot 4 + 9}{4^3}$

$$= \frac{37}{64} \approx 0.578125 \approx 58\%.$$
Let \( X = \) amount won or lost.

\[
E[X] = (\$5)\left(\frac{37}{64}\right) + (-\$6)\left(\frac{27}{64}\right)
\]

probability that 3 is rolled within the first three rolls

\[
= \frac{\$185 - 162}{64} = \frac{\$23}{64} \approx \$0.359...
\]

So you can expect to win on average about 36 cents per play. Yes, take the bet.

\[ S = \{1, 2, 3, 4\} \]

\[ P(1) = \frac{2}{8}, \quad P(2) = \frac{1}{8}, \quad P(3) = \frac{3}{8}, \quad P(4) = \frac{2}{8} \]

(a) Let \( X = \) amount won or lost

\[
E[X] = (\$2) P(1) + (\$2) P(2) + (-\$1) P(3) + (-\$1) P(4)
\]

\[
= (\$2)\left(\frac{2}{8}\right) + (\$2)\left(\frac{1}{8}\right) + (-\$1)\left(\frac{3}{8}\right) + (-\$1)\left(\frac{2}{8}\right)
\]

\[
= \$(\frac{1}{8}) \approx \$0.125.
\]

(b) Let \( a \) be the amount won or lost when 1 is rolled,

\( b \) 2 is rolled,

\( c \) 3 is rolled,

\( d \) 4 is rolled.

Let \( Y \) be the total amount won or lost.
Then
\[ E[\Xi] = a\left(\frac{2}{8}\right) + b\left(\frac{1}{8}\right) + c\left(\frac{3}{8}\right) + d\left(\frac{2}{8}\right) \]
\[ = \frac{2a + b + 3c + 2d}{8} \]

For \( E[\Xi] = 0 \) we would need \[ 2a + b + 3c + 2d = 0 \]

There are infinitely many solutions to this equation. 
For example, set \( a = 1, b = 1, c = -1, d = 0 \). 
There are many other solutions.

There are 47 cards left in the deck.

(a) Out of the 47 cards left there are 13 - 3 = 10 hearts left. The odds of getting 2 more hearts is
\[ \frac{\binom{10}{2}}{\binom{47}{2}} = \frac{\frac{10!}{8!2!}}{\frac{47!}{45!2!}} = \frac{\binom{10}{2}}{\binom{47}{2}} = \frac{45}{1081} \approx 0.0416 \approx 4\% \]

(b) Choose a heart (47 - 10 = 37)
\[ \frac{10 \cdot 37}{47} = \frac{370}{1081} \approx 0.34 \approx 34\% \]

(c) There are 3 queens left and 47 - 3 = 44 non-queens left. Answer
\[ \frac{3 \cdot 44}{\binom{47}{2}} = \frac{132}{1081} \approx 0.122... \]
choose 2 queens out of the 3 queen left
i.e. number of ways to get exactly 2 queens

\[
\binom{3}{2} = \frac{3}{\binom{4}{2}} = \frac{3}{1081} \approx 0.00277521\
\]

(e) This happens if you get exactly one queen or exactly two queens.
Add (c) and (d) to get

\[
\frac{132}{1081} + \frac{3}{1081} = \frac{135}{1081} \approx 0.12488 \times 12.9
\]

(f) Let \( X = \) amount won or lost.

Then

\[
E[X] = (\$500 \text{ probability you get a flush}) + (-\$20 \text{ probability you don't get a flush})
\]

\[
= (\$500 \cdot \frac{45}{1081}) + (-\$20 \cdot \left(1 - \frac{45}{1081}\right))
\]

from (a)

\[
= \$ \left(\frac{22,500 - 20,720}{1081}\right) = \$ \left(\frac{1750}{1081}\right)
\]

\[\approx \$1.6466\]

Well, it's a positive expected value. If you were in a casino this is the kind of game you would want people to be able to play against you.
5) From hw #1 we have the sample space

\[ S = \{ (H), (T,H), (T,T,H), (T,T,T,H), \ldots \} \]

and

\[ P(\{(T,T,T,\ldots ,T,H)\}) = \left( \frac{1}{2} \right)^{k+1} \]

Let \( E = \{ (T,T,T,H), (T,T,T,T,H), (T,T,T,T,T,H), \ldots \} \)

We want \( P(E) \). Here \( E \) is the event that you win the bet and \( \overline{E} \) is the event that you lose the bet.

**Direct method:**

\[
P(E) = \left( \frac{1}{2} \right)^4 + \left( \frac{1}{2} \right)^5 + \left( \frac{1}{2} \right)^6 + \ldots = \left( \frac{1}{2} \right)^4 \left[ \frac{1}{1-\frac{1}{2}} \right] = \frac{1}{8}
\]

**Indirect method**

\[
P(E) = 1 - P(\overline{E}) = 1 - P(\{(H), (T,H), (T,T,H)\}) = 1 - \frac{1}{2} - \frac{1}{2^2} - \frac{1}{2^3} = \frac{8 - 4 - 2 - 1}{8} = \frac{1}{8}
\]

So \( P(E) = \frac{7}{8} \).

Let \( X \) = amount won or lost if you take the bet.

Then

\[
E[X] = (\$5) \left( \frac{1}{8} \right) + (-\$1) \left( \frac{7}{8} \right) = -\$\frac{2}{8} = -\$0.25
\]

You expect to lose \$0.25 each time you play on average if you play many times.