How do we generalize this?

Suppose we want to do two experiments, one after the other and the influence of the outcome of the first does not affect the outcome of the other.

Let \((S_1, \Omega_1, P_1)\) and \((S_2, \Omega_2, P_2)\) be probability spaces corresponding to each experiment. Define the new space \((S, \Omega, P)\) where \(S = S_1 \times S_2\) and \(\Omega\) is the smallest \(\sigma\)-algebra containing all subsets of \(S\) of the form \(A_1 \times A_2\) where \(A_i \in \Omega_i\).

A probability function \(P\) is defined on each outcome \((w_1, w_2) \in S_1 \times S_2\) as

\[
P(\{(w_1, w_2)\}) = P_1(\{w_1\}) \cdot P_2(\{w_2\}),
\]

If \(S\) is finite then we extend the definition to any events \(E \times F \subseteq S_1 \times S_2\) as follows:

\[
P(E \times F) = \sum_{(e,f) \in E \times F} P(e,f) = \sum_{e \in E} \sum_{f \in F} P_1(e)P_2(f)
\]

\[
= \left(\sum_{e \in E} P_1(e)\right) \cdot \left(\sum_{f \in F} P_2(f)\right) = P_1(E) \cdot P_2(F).
\]

That is

\[
P(E \times F) = P_1(E) \cdot P_2(F).
\]

Note that \(P(S) = P(S_1 \times S_2) = P_1(S_1) \cdot P_2(S_2) = 1 \cdot 1 = 1\), this construction is called the product space.
For Tony:

\[ P(S) = P(S_1 \times S_2) = \sum_{(a,b) \in S_1 \times S_2} P(a,b) = \sum_{\text{aes}_i} \sum_{\text{bes}_2} P_i(a)P_2(b) \]

\[ = \left( \sum_{\text{aes}_i} P_i(a) \right) \left( \sum_{\text{bes}_2} P_2(b) \right) = 1 \cdot 1 = 1 \]