How do we generalize this?

Suppose we want to do two experiments, one after the other and the outcome of either experiment does not influence the outcome of the other.

Let (S_1, Ω_1, P_1) and (S_2, Ω_2, P_2) be probability spaces corresponding to each experiment. ~~Suppose Ω_1 and Ω_2 consist of all the subsets of S_1 and S_2, respectively.~~

Define the new space (S, Ω, P) where $S = S_1 \times S_2$, Ω be the smallest σ-algebra containing all subsets of S of the form $A_1 \times A_2$ where $A_1 \in \Omega_1$ and $A_2 \in \Omega_2$.

and P is defined on each outcome $(w_1, w_2) \in S_1 \times S_2$ as

$$\boxed{P(\{(w_1, w_2)\}) = P_1(\{w_1\}) \cdot P_2(\{w_2\})}.$$

If S is finite then we extend the def as follows: to any events $E \times F \subseteq S_1 \times S_2$

$$P(E \times F) = \sum_{(e,f) \in E \times F} P(e,f) = \sum_{e \in E} \sum_{f \in F} P_1(e) P_2(f)$$

$$= \left[\sum_{e \in E} P_1(e)\right] \cdot \left[\sum_{f \in F} P_2(f)\right] = P_1(E) \cdot P_2(F).$$

That is $\boxed{P(E \times F) = P_1(E) \cdot P_2(F)}$

Note that $P(S) = P(S_1 \times S_2) = P_1(S_1) \cdot P_2(S_2) = 1 \cdot 1 = 1$.

This construction is called the product space.

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For Tony:

$$P(S) = P(S_1 \times S_2) = \sum_{(a,b)\in S_1\times S_2} P(a,b) = \sum_{\substack{a\in S_1\\ b\in S_2}} P_1(a)P_2(b)$$

$$= \left(\sum_{a\in S_1} P_1(a)\right)\left(\sum_{b\in S_2} P_2(b)\right) = 1\cdot 1 = 1$$