Let's calculate the probability of winning.

\[
P(\text{win}) = \frac{8}{36} + \frac{3}{39} \cdot \frac{3}{9} + \frac{4}{36} \cdot \frac{4}{10} + \frac{5}{36} \cdot \frac{5}{11} + \frac{5}{36} \cdot \frac{5}{11}
\]

Point is 4 Point is 5 Point is 6 Point is 8

\[
\frac{4}{36} \cdot \frac{4}{10} + \frac{3}{36} \cdot \frac{3}{9} = \frac{244}{495} \approx 0.4929
\]

Point is 9 Point is 10

\[
P(\text{lose}) = 1 - P(\text{win}) = 1 - \frac{244}{495} = 0.5071
\]

Expected Value

Suppose you put $1 on the pass line. Let \( X \) be the amount of money won or lost when you play one round of craps.

\[
E[X] = (\$1)(P(\text{win})) + (-\$1)(P(\text{lose}))
\]

\[
= (\$1)(\frac{244}{495}) + (-\$1)(\frac{251}{495}) \approx -$0.0141...
\]

\[
\approx -1.41\text{ cents.}
\]

Video lose 1.41 cents per $1

Real odds

Casino pays 1-1 which gives you expected value of -$0.014.

The true odds are

\[
\text{odd of losing} = \frac{P(\text{lose})}{P(\text{win})} = \frac{251}{244} \times \frac{244}{244} = \frac{251}{244} \approx 1.04149378...
\]

The game would be "fair" if the casino paid \( \frac{251}{244} : 1 \)
If they paid $\frac{251}{244}$ when you win the expected value would be

$$E(X) = (\frac{251}{244}) \cdot \left(\frac{244}{434}\right) + (-1) \cdot (\frac{251}{434}) = \$0$$

By "fair" game we mean expected value is $\$0$.

The casino allows an extra "free odds" bet if a point is made. The free odds bets are paid off at the true odds. Thus making them a fair bet, that the casino has no edge on true bets.

<table>
<thead>
<tr>
<th>Point</th>
<th>True Odds</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>2:1</td>
</tr>
<tr>
<td>5</td>
<td>3:2</td>
</tr>
<tr>
<td>6</td>
<td>6:5</td>
</tr>
<tr>
<td>8</td>
<td>6:5</td>
</tr>
<tr>
<td>9</td>
<td>3:2</td>
</tr>
<tr>
<td>10</td>
<td>2:1</td>
</tr>
</tbody>
</table>

The odds against 4 win = \[\frac{\text{Prob 7 rolled before 4}}{\text{Prob 4 rolled before 7}} = \frac{6/9}{3/9} = \frac{2}{1}\]

Example: Suppose you bet $\$10$ on the pass line. Suppose a 5 is rolled as the point.

Free odds $\$50$

If you win you get

$$($$10)(\frac{1}{3}) + ($$50)(\frac{3}{2}) = $$10 + $$75 = $$85

If you lose you lose $\$60$.
Ex: Suppose you have a box with 3 red balls and 10 black balls. Suppose you draw one ball and look at the color and don't put it back in the box. Then you draw another ball.

Let F be the event that the first ball is red. Let E be the event that the second ball is red. Are these independent events?

E and F are independent, if

\[ P(E \cap F) = P(E)P(F) \]

* is equivalent to ** and
*** if \( P(E) \neq 0 \) and \( P(F) \neq 0 \)

**

\[ P(E \mid F) = P(E) \quad \text{and} \quad P(F \mid E) = P(F) \]

\[ P(F) = \frac{3}{13} \]

\[ P(E) = P(E \mid \text{first ball is red})P(\text{first ball is red}) + P(E \mid \text{first ball is black})P(\text{first ball is black}) \]

\[ P(E) = \frac{2}{12} \cdot \frac{3}{13} + \frac{3}{12} \cdot \frac{10}{13} = \frac{36}{156} \]

\[ P(E \cap F) = P(\text{1st ball is red} \Rightarrow F) = \frac{3 \cdot 2}{15 \cdot 12} = \frac{6}{156} \approx 0.03846 \]

They are not independent.
Suppose we bet $10 on the pass line and if a point is made then we bet $10 more on the free odds bet. What’s the expected value of this strategy.

\[
\begin{array}{cc}
\text{Point} & \text{True odds} \\
4 or 10 & 2:1 \\
5 or 9 & 3:2 \\
6 or 8 & 6:5 \\
\end{array}
\]

\[
\begin{array}{cc}
\text{Point} & \text{Probability of point before 7} & \text{Probability of 7 before point} \\
4 or 10 & 3/9 & 6/9 \\
5 or 9 & 4/10 & 6/10 \\
6 or 8 & 5/11 & 6/11 \\
\end{array}
\]

Free odds $10

True odds $12

\[
\begin{array}{c}
\text{Pass line} \\
\text{Pass} \\
\text{we win} \quad $10 + $10(1/2) = $10 + $12 = $22 \\
\end{array}
\]

\[
\begin{array}{cccc}
1\text{st roll} & 2\text{nd roll} & 3\text{rd roll} & 4\text{th roll} \\
8 & 6 & 4 & 8 \\
\uparrow \text{point} & & & \text{we win} \\
\end{array}
\]

\[
\text{we win} \quad $10 + $10(1/2) = $10 + $12 = $22 \\
\text{true odds}
\]
Expected value = \( (\$10) \cdot \left( \frac{8}{36} \right) + (-\$10) \left( \frac{4}{36} \right) \)

\[ \begin{align*}
\text{first roll is 7 or 11} & \text{ first roll is 2, 3, or 12, lose} \\
\left( \frac{3}{36} \right) \left( \frac{2}{9} \right) \left( \$30 \right) & + \left( \frac{3}{36} \right) \left( \frac{6}{9} \right) (-\$20) & \left( \frac{3}{36} \right) \left( \frac{3}{4} \right) \left( \$30 \right) \\
\text{Point is 4 we win} & \text{ Point is 4 and we lose} & \text{Point is 10 and we win} \\
\left( \frac{3}{36} \right) \left( \frac{5}{9} \right) (-\$20) & + 2 \left( \frac{4}{36} \right) \left( \frac{4}{10} \right) \left( \$25 \right) & + 2 \left( \frac{4}{36} \right) \left( \frac{6}{10} \right) (-\$20) \\
\text{Point is 5 or 9 and we win} & \text{ Point is 5 or 9 and we lose} & \\
2 \left( \frac{5}{36} \right) \left( \frac{5}{11} \right) \left( \$22 \right) & + 2 \left( \frac{5}{36} \right) \left( \frac{6}{11} \right) (-\$20) & = \$ \frac{14}{99} \approx \$0.1414 \\
\text{Point is 6 or 8 and we win} & \text{ Point is 6 or 8 and we lose} \\
\end{align*} \]

Let's put this in "per $1" terms.

average amount wagered = \( \frac{1}{3} \left( \$10 \right) + \frac{2}{3} \left( \$20 \right) \approx \$16.67 \)

\( \frac{12}{36} = \frac{1}{3} \) 1st roll is 7, 11, 2, 3, 12 we only bet \$10
\( \frac{12}{36} = \frac{1}{3} \) point is made we placed an extra \$10 down

expected value per dollar wagered \( \approx \frac{-\$0.1414}{\$16.67} \approx -\$0.0085 \)

Last time with betting \$1 on pass line expected value \( \approx -\$0.014. \)
How much should you pay to play this game?
Pot starts at $2.
the first time a tail is flipped you win the pot and the game is over. Every time heads is flipped the pot doubles.

$$\text{Pay} = \boxed{\$4}$$

| Pot   | Flip | Lost $2 |  \\
<table>
<thead>
<tr>
<th></th>
<th></th>
<th></th>
</tr>
</thead>
<tbody>
<tr>
<td>$2</td>
<td></td>
<td>T</td>
</tr>
<tr>
<td>$4</td>
<td>H</td>
<td></td>
</tr>
<tr>
<td>$8</td>
<td>H</td>
<td></td>
</tr>
<tr>
<td>$16</td>
<td>T</td>
<td></td>
</tr>
</tbody>
</table>

$$\text{Total Win} = 16 - 4 = \$12$$

$$\text{Expected value} = (-\#M) + (\#2)(\frac{1}{2}) + (\#4)(\frac{1}{2} \cdot \frac{1}{2}) + \$8 \left(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}\right) + (\#16)(\frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2} \cdot \frac{1}{2}) + \ldots$$

$$= \$ \left[-M + 1 + 1 + 1 + 1 + \ldots\right] = \$\infty$$