Probability function for independent compound experiments

**Example:** Suppose you flip a fair coin then you roll a 4-sided die, what is the probability that you get a heads on the coin followed by a 2 on the die?

\[
\frac{1}{2} \cdot \frac{1}{4} = \frac{1}{8}
\]

All outcomes here are equally likely.

How to generalize this?

Suppose we want to do two experiments, in a row where the outcome of the first experiment does not influence the second experiment.

Let \((S_1, \Omega_1, P_1)\) and \((S_2, \Omega_2, P_2)\) be probability spaces corresponding to the two experiments.

Define \((S, \Omega, P)\) for the compound experiment as follows:

\[S = S_1 \times S_2\]

the smallest \(\sigma\)-algebra containing all subsets of \(S\) of the form \(E_1 \times E_2\) where \(E_1 \in \Omega_1\) and \(E_2 \in \Omega_2\)

\(P\) is defined as \(P((w_1, w_2)) = P_1(w_1) \cdot P_2(w_2)\) \(\forall w_1 \in S_1\), \(w_2 \in S_2\)

Then extend \(P\) where \(P(E) = \sum P(e)\), \(e \in E\)

then \((S, \Omega, P)\) will be a probability space.
Example: Suppose you have a weighted 4-sided die and a fair coin. Model the compound experiment where we roll the die and flip the coin, where \( \hat{p}(1) = \frac{1}{8}, \hat{p}(2) = \frac{1}{4}, \hat{p}(3) = \frac{1}{2}, \hat{p}(4) = \frac{1}{8} \)

\[
S_1 = \{1, 2, 3, 4\} \\
S_2 = \{H, T\}
\]

\[
P_1(1) = \frac{1}{8} \\
P_1(2) = \frac{1}{4} \\
P_1(3) = \frac{1}{2} \\
P_1(4) = \frac{1}{8}
\]

\[
P(H) = \frac{1}{2} \\
P(T) = \frac{1}{2}
\]

\(\Omega_1 = \text{all subsets of } S_1\)

\(\Omega_2 = \text{all subsets of } S_2\)

\(S = S_1 \times S_2 = \{1H, 1T, 2H, 2T, 3H, 3T, 4H, 4T\}\)

\(\Omega = \text{all subsets of } S\)

\[
P([1H]) = \hat{p}_1(1) \times \hat{p}_2(H) = \frac{1}{8} \times \frac{1}{2} = \frac{1}{16}
\]

Justification: We can model this with an equally weighted one-pretend that you have an 8-sided die that is fair with sides label 1, 2, 2, 3, 3, 3, 3, 4 \( \frac{1}{8} \times \frac{1}{2} = \frac{1}{16} \)
HW #2

(a) Suppose we have 5 math books, 3 bio books, 2 history books, and 3 literature books. How many ways can we put the books on a shelf?

(b) How many ways where the math books have to be together?

(a) 13 book so it is 13!

(b) \[ b_1 \ b_2 \ b_3 \ h_1 \ h_2 \ l_1 \ l_2 \ l_3 \ \text{math} \]

9! ways to arrange those objects.

For each of these 9! ways we can arrange the math books in 5! ways.

Answer = (9!)(5!)
Conditional Probability

Example: Suppose we toss two six-sided dice, a green die and a red die. Suppose when you roll them the green die stops first and lands on a 3 but the red one hasn’t stopped rolling yet. What is the probability that the sum of the two dice will be 8?

Original Sample Space (Rolling 2 dice)

\[
S = \{(1,1), (1,2), (1,3), \ldots, (1,6), (2,1), \ldots, (3,1), \ldots, (6,1), \ldots, (6,6)\}
\]

Since you know that the green die landed on a 3 it shrinks the sample space

\[
\text{(green=3) new Sample Space } S' = \{(1,3), (2,3), (3,3), (4,3), (5,3), (6,3)\}
\]

Probability = \(\frac{1}{6}\)
Let's make a formula that doesn't shrink the sample space and will generalize to spaces where the outcomes are not equally likely.

Let $S = \{(i, 19) | 1 \leq i, 9 \leq 6, i, 9 \in \mathbb{Z}_6\}$

Let $E$ be the event in $S$ where the sum of the dice is 8, and $F$ be the event where the green die is 3. We want to know the "conditional probability" of the event $E$ occurring given that $F$ has already occurred.

\[
\frac{|E \cap F|}{|F|} = \frac{1}{6}
\]

\[
\frac{|E \cap F|}{|F|} = \frac{|E \cap F|}{|F|} \leq \frac{1}{6}
\]

\[
P(E|F) = \frac{|E \cap F|}{|F|} = \frac{1}{6}
\]

only works because every outcome is equally likely.
Def: Let $(S, \Omega, P)$ be a probability space.

- Let $E$ and $F$ be events with $P(F) > 0$ we define the conditional probability that $E$ occurs given that $F$ occurred as

$$P(E|F) = \frac{P(E \cap F)}{P(F)}$$

Example: Let $S, E, F$ be as in previous example.

$$P(E|F) = \frac{P(E \cap F)}{P(F)} = \frac{\left(1/\frac{3b}{b}\right)}{\left(\frac{1}{3b}\right)} = \frac{1}{b}$$

Some Properties: Let $(S, \Omega, P)$ be a probability space

1. Let $A$ and $B$ be events with $P(A) > 0$ then $P(AB) = P(A)P(B|A)$

2. Let $A_1, A_2, A_3, \ldots, A_n$ be events with

$$P\left(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n\right) > 0$$

then

$$P\left(A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_n\right) = P(A_1) P(A_2|A_1) P(A_3|A_1 \cap A_2) \cdots P(A_n|A_1 \cap A_2 \cap A_3 \cap \ldots \cap A_{n-1})$$

3. (Law of total probability)

Suppose that $S = E_1 \cup E_2 \cup \ldots \cup E_n$ where each $E_i \neq \emptyset$ and $E_i \cap E_j = \emptyset$ if $i \neq j$.

Further suppose that $P(E_i) > 0 \ \forall i$, then if $E$ is any event then

$$P(E) = P(E \cap E_1) + P(E \cap E_2) + \cdots + P(E \cap E_n)$$

Proof of (3):

$$P(E) = P(E \cap E_1) + P(E \cap E_2) + \cdots + P(E \cap E_n)$$

axiom $6 = P(E \cap E_1)P(E_1) + \cdots + P(E \cap E_n)P(E_n) < Cond. \ Prob.$
Example: Suppose there are 3 boxes and in
box 1 there are 2 4-sided dice \( (1,4) \)
box 2 there are 2 6-sided dice \( (1,6) (3,5) (6,2) (3,3) (1,4) \)
box 3 there are 2 8-sided dice \( (1,7) (1,6) (3,5) (1,4) \)
\( (3,3) (4,2) (7,1) \)
suppose you randomly pick a box, each box is
equally likely and then you pull out the dice
from the box you choose and roll them.

What is the probability the sum is 8.

* Use Law of Total Probability

\[
P(\text{sum is 8}) = P(\text{sum = 8 | box 1 chosen}) \cdot P(\text{box 1 chosen}) + P(\text{sum = 8 | box 2 chosen}) \cdot P(\text{box 2 chosen}) + P(\text{sum = 8 | box 3 chosen}) \cdot P(\text{box 3 chosen})
\]

\[
= \left( \frac{1}{16} \right) \left( \frac{1}{3} \right) + \left( \frac{5}{36} \right) \left( \frac{1}{3} \right) + \left( \frac{7}{64} \right) \left( \frac{1}{3} \right)
\]

\[
= \frac{11,456}{110,1592} \approx 0.1036
\]

Monte Hall:

1: goat
2: car
3: goat