Some properties

Let (S, Ω, P) be a probability space.

① Let A and B be events with $P(A) > 0$. Then

$$P(A \cap B) = P(A)\,P(B|A)$$

② Let $A_1, \dots, A_n$ be events with $P(A_1 \cap A_2 \cap \dots \cap A_n) > 0$ then

$$P(A_1 \cap A_2 \cap \dots \cap A_n) = P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2) \cdot P(A_4|A_1 \cap A_2 \cap A_3) \cdots P(A_n|A_1 \cap A_2 \cap \dots \cap A_{n-1})$$

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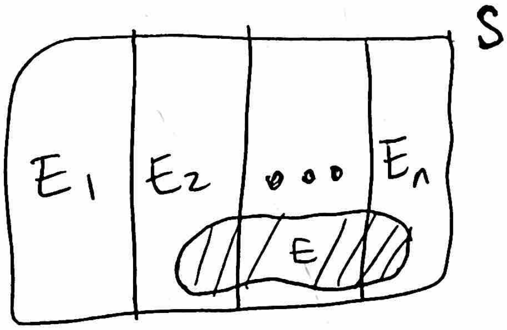
③ (Law of total probability) Suppose that $S = E_1 \cup E_2 \cup \dots \cup E_n$ where each E_i is non-empty and $E_i \cap E_j = \emptyset$ if $i \neq j$ (ie each of the E_i and E_j are disjoint). Suppose further that $P(E_i) \neq 0$ for all i.



Then if E is any event then

$$P(E) = P(E|E_1) \cdot P(E_1) + P(E|E_2) \cdot P(E_2) + \dots + P(E|E_n) \cdot P(E_n)$$

proof: ① From def of $P(B|A) = \frac{P(A \cap B)}{P(A)}$.

② Here it is for 3 sets A_1, A_2, and A_3: $P(A_1 \cap A_2 \cap A_3) = P((A_1 \cap A_2) \cap A_3)$

$$\overset{①}{=} P(A_1 \cap A_2) \cdot P(A_3|A_1 \cap A_2) \overset{①}{=} P(A_1) \cdot P(A_2|A_1) \cdot P(A_3|A_1 \cap A_2).$$

③ Note $P(E) = P((E \cap E_1) \cup (E \cap E_2) \cup \dots \cup (E \cap E_n)) \overset{\text{axiom } 6}{=} \sum_{i=1}^{n} P(E \cap E_i)$

$$= \sum_{i=1}^{n} P(E|E_i) \cdot P(E_i)$$