Theorem

Let S be a sample space of a repeatable experiment. Let A and B be mutually exclusive events in S [ie $A \cap B = \phi$]. Suppose further that each time we repeat S the experiment is independent of the previous experiments. Suppose we repeat S until either A or B occurs. Then the probability that A occurs before B is $\frac{P(A)}{P(A)+P(B)}$.

Ex: Suppose we roll two 6-sided dice over and over. Let A be the event that the sum of the dice is 4. Let B be the event that the sum of the dice is 7. The probability that A occurs before B is

$$\frac{P(A)}{P(A)+P(B)} = \frac{3/36}{3/36 + 6/36} = \frac{3}{9} = 1/3$$

The probability that B occurs before A is

$$\frac{P(B)}{P(B)+P(A)} = \frac{6/36}{6/36+3/36} = \frac{6}{9} = 2/3$$

proof of theorem:

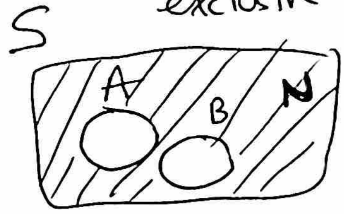
Let E be the event that A occurs before B.
Let A_1, B_1, N_1 be the events that A occurs on the first experiment, B occurs on the first experiment, or neither occurs on the first experiment.

Then

$$P(E) = P(E|A_1)P(A_1) + P(E|B_1)P(B_1) + P(E|N_1)P(N_1)$$

$$= 1 \cdot P(A) + 0 \cdot P(B) + P(E|N_1) \cdot \underbrace{[1 - P(A) - P(B)]}_{\text{because mutually exclusive}}$$

$$= P(A) + P(E)[1 - P(A) - P(B)]$$



$P(E|N_1) = P(E)$
since the outcomes of successive experiments are all independent of each other, when the second experiment begins, the whole procedure probabilistically starts over again. Therefore, if in the 1st experiment neither A nor B occurs, the probability of E before doing the 1st experiment and after doing the 1st experiment is the same

Thus,

$$P(E) - P(E)[1 - P(A) - P(B)] = P(A)$$

So,

$$P(E) = \frac{P(A)}{P(A) + P(B)}$$