Math 4680 - Homework # 8

Sequences

1. Consider the sequence $(z_n)_{n=1}^{\infty}$ where $z_n = \dfrac{1}{n} + \dfrac{(n-1)i}{n}$.

   (a) Use the definition of limit to show that $\lim_{n\to\infty} z_n = i$.

   (b) Use the theorem from class to break the above into real valued limits and use calculus to show that $\lim_{n\to\infty} z_n = i$.

2. Consider the sequence $(z_n)_{n=1}^{\infty}$ where $z_n = -2 + i\dfrac{(-1)^n}{n^2}$.

   (a) Use the definition of limit to show that $\lim_{n\to\infty} z_n = -2$.

   (b) Use the theorem from class to break the above into real valued limits and use calculus to show that $\lim_{n\to\infty} z_n = -2$.

3. Let $(z_n)_{n=1}^{\infty}$ be a sequence of complex numbers where $z_n = x_n + iy_n$ with $x_n, y_n \in \mathbb{R}$ for all n. Prove that (z_n) is a Cauchy sequence in the complex numbers if and only if both (x_n) and (y_n) are Cauchy sequences in the real numbers. :

4. Let $(z_n)_{n=1}^{\infty}$ be a sequence of complex numbers. Prove: If (z_n) converges, then (z_n) is bounded. (By bounded we mean that there exists $M > 0$ such that $|z_n| \leq M$ for all n.)

5. Let $(z_n)_{n=1}^{\infty}$ and $(w_n)_{n=1}^{\infty}$ be sequences of complex numbers. Suppose that $\lim_{n\to\infty} z_n = A$ and $\lim_{n\to\infty} w_n = B$. Prove:

   (a) If $\alpha, \beta \in \mathbb{C}$, then $\lim_{n\to\infty} \alpha z_n + \beta w_n = \alpha A + \beta B$

   (b) $\lim_{n\to\infty} z_n w_n = AB$

6. Let $F \subseteq \mathbb{C}$. Prove that F is a closed set if and only if whenever $(z_n)_{n=1}^{\infty}$ is a sequence of points in F such that $w = \lim_{n\to\infty} z_n$ exists, then $w \in F$.

THE NEXT PROBLEM ISN'T NECESSARY TO DO. Just do it if you feel like it, or read the solutions to see how the proof goes if interested.

7. Let γ be a curve. That is, $\gamma : [a, b] \to \mathbb{C}$ where $\gamma(t) = u(t) + iv(t)$ where u and v are continuous on $[a, b]$. Prove that the the image $\gamma([a, b])$ is a closed set in $\mathbb{C}$.