Math 4680 - Homework # 1

Complex numbers

1. For each complex number z, do the following: graph z, calculate $\overline{z}$, graph $\overline{z}$, and calculate $|z|$.

   (a) $z = 1 + i$

   (b) $z = -1 - 3i$

   (c) $z = \frac{1}{2} - \pi i$

2. Express the following complex numbers in the form $a + bi$.

   (a) $\dfrac{2+3i}{4+i}$

   (b) $(\sqrt{2} - i)(1 - i\sqrt{2})$

   (c) $\dfrac{1+2i}{3-4i} + \dfrac{2-i}{5i}$

   (d) $(1-i)^4$

   (e) $\left(2 + \dfrac{1}{1-i}\right)^2$

3. Find the absolute value of the following complex numbers.

   (a) $\dfrac{i(2+4i)(1-2i)}{(2-i)}$

   (b) $\dfrac{(3i)^2}{(-3+i)^6}$

4. For each pair $z_1, z_2 \in \mathbb{C}$ do the following: (i) write each element in polar form and graph the polar coordinates, (ii) compute the polar form of $z_1 \cdot z_2$ and graph it.

   (a) $z_1 = 1 + i$ and $z_2 = \overline{z_1} = 1 - i$.

   (b) $z_1 = 1 + i$ and $z_2 = -1$

5. Solve the following equations.

(a) $z^2 - i = 0$
(b) $z^4 + i = 0$
(c) $z^6 = -64$
(d) $z^3 + (1 + i) = 0$

6. Describe and sketch each of the following sets of complex numbers.

   (a) $S = \{z \in \mathbb{C} \mid \text{Im}(z + 5) = 0\}$
   (b) $S = \{z \in \mathbb{C} \mid |z^2| \geq 4\}$
   (c) $S = \{z \in \mathbb{C} \mid |z - 2 + i| \leq 3\}$
   (d) $S = \{z \in \mathbb{C} - \{0\} \mid \text{Re}(1/z) \geq 1/2\}$

7. Find the real and imaginary parts of the following where $z = x + iy$

   (a) $\dfrac{1}{z^2}$
   (b) $\dfrac{z - 1}{3z + 2}$

8. Prove the following for $z, w \in \mathbb{C}$.

   (a) $\overline{z + w} = \overline{z} + \overline{w}$
   (b) $\overline{zw} = \overline{z} \cdot \overline{w}$
   (c) $|z|^2 = z\overline{z}$
   (d) $|zw| = |z||w|$
   (e) $\left|\dfrac{z}{w}\right| = \dfrac{|z|}{|w|}$ if $w \neq 0$.
   (f) Show that $\text{Re}(iz) = -\text{Im}(z)$ and that $\text{Im}(iz) = \text{Re}(z)$.

9. Prove: For all $z_1, z_2, z_3, z_4 \in \mathbb{C}$ with $|z_3| \neq |z_4|$ we have that

$$\left|\frac{z_1 + z_2}{z_3 + z_4}\right| \leq \frac{|z_1| + |z_2|}{||z_3| - |z_4||}$$

10. Prove the following.

   (a) Let $z \in \mathbb{C}$. z is real if and only if $\overline{z} = z$.

(b) Let $z \in \mathbb{C}$. z is either real or pure imaginary if and only if $(\overline{z})^2 = z^2$.

11. Let $n \geq 2$ be an integer. Let $w \in \mathbb{C}$ be an n-th root of unity (that is $w^n = 1$) with $w \neq 1$. Prove that $1 + w + w^2 + \cdots + w^{n-1} = 0$

THE NEXT TWO AREN'T NECESSARY TO DO. Just do them if you feel like it, or read the solutions to see how the proofs go.

A. (De Moivre's Formula) If $z = r[\cos(\theta) + i\sin(\theta)]$ and n is a positive integer, then $z^n = r^n[\cos(n\theta) + i\sin(n\theta)]$.

B. Let $w = r[\cos(\theta) + i\sin(\theta)]$ where $w \neq 0$. The n-th roots of w, that is the solutions to $z^n = w$, are given by

$$z_k = r^{1/n}\left[\cos\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)\right]$$

where $k = 0, 1, 2, \ldots, n-1$.