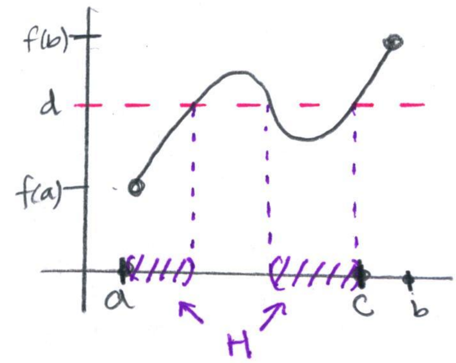
Recap from last time

Intermediate Value Thm: Let f be continuous on $[a,b]$, and $f(a) < f(b)$ given d with $f(a) < d < f(b)$. There exists c with $a<c<b$ and $f(c)=d$



proof:

let $H = \{x \mid a<x<b \text{ and } f(x)<d\}$

We showed (1) $H \neq \emptyset$

(2) b is an upper bound for H

so, $\sup(H)$ exists. Let $c = \sup(H)$

Note: Last thing we showed was $c<b$

Note that $a<c$

Why? since $H \neq \emptyset$, $\exists\, h \in H$

since $h \in H$, we have $a<h$

since $c = \sup(H)$, we have $h \leq c$ so, $a<h\leq c$.

We now need to show that $f(c)=d$

We do this by showing that $f(c)>d$ and $f(c)<d$ cannot happen

- Suppose $f(c)<d$

Then $d-f(c)>0$. Let $\varepsilon = d-f(c)$

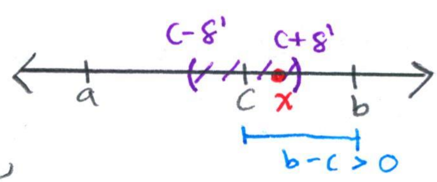
since $a<c<b$, we know that f is continuous at c.

so there exists $\delta'>0$, where if $x\in[a,b]$ and $|x-c|<\delta'$

then $|f(x)-f(c)| < \varepsilon = d-f(c)$

* Always true $y \leq |y|$

Let us assume $\delta' < b-c$



so if $x\in[a,b]$ and $|x-c|<\delta'$ then,

$f(x)-f(c) \leq |f(x)-f(c)| < d-f(c)$

so for $x\in[a,b]$ with $|x-c|<\delta'$ we have $f(x)<d$

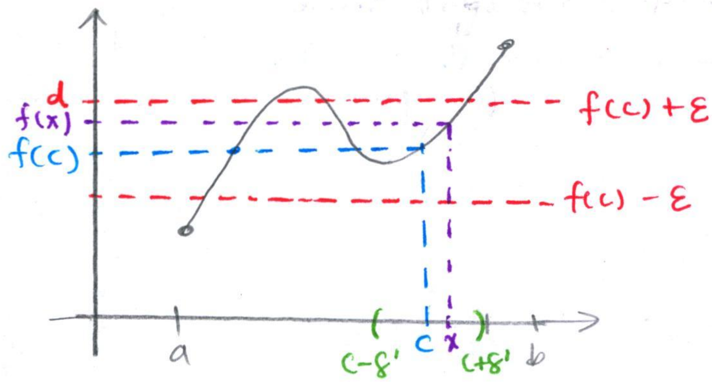
that is $x\in H$

For example if $x = c + \frac{1}{2}\delta'$,

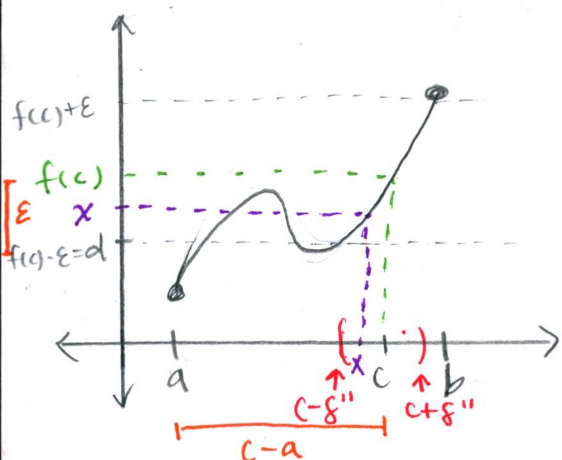
then $c < x < b$ and $f(x) < d$

then $x \in H$ and contradicts

the fact that $c = \sup(H)$.



- Suppose $f(c) > d$



Let $\varepsilon = f(c) - d > 0$

since f is continuous at c $\exists\ \delta'' > 0$

where if $x \in [a,b]$ and $|x-c| < \delta''$ then

$|f(x) - f(c)| < \varepsilon = f(c) - d$

- we may assume $\delta'' < c - a$

so if $x \in [a,b]$ and $|x-c| < \delta''$ then $f(c) - f(x) \leq |f(x) - f(c)| < f(c) - d$

That is, if $x \in [a,b]$ and $|x-c| < \delta''$, then $d < f(x)$.

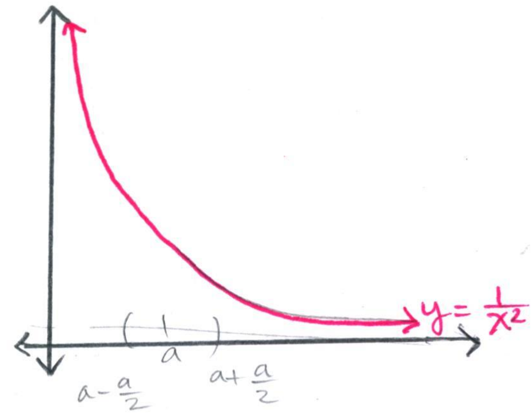
so, $(c-\delta'', c+\delta'') \cap H = \emptyset$.

However, by the useful sup fact, there must exists

$h \in H$ with $c - \delta'' < h < c$.

contradiction! $\boxtimes$

Homework 4 #4



Let $f(x) = \frac{1}{x^2}$, let $a \in \mathbb{R}$, $a \neq 0$.

Show that f is continuous at a.

Proof: Let's assume $a > 0$

(A similar proof will work for $a < 0$)

Let $\varepsilon > 0$,

we need to find $\delta > 0$ where if $x \neq 0$ (ie. x is in the domain of f) and $|x-a| < \delta$ then $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < \varepsilon$.

Note that,

control via δ

need to bound by a number

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \left|\frac{a^2 - x^2}{x^2 a^2}\right| = \frac{|a-x||a+x|}{|x|^2\,|a|^2} = \frac{\boxed{|x-a|}\,|x+a|}{|x|^2\,|a|^2}$$

Suppose $\delta \leq \frac{a}{2}$

Suppose $|x-a| < \delta < \frac{a}{2}$ so, $a - \frac{a}{2} < x < a + \frac{a}{2}$. That is, $\frac{a}{2} < x < \frac{3a}{2}$.

So, $\frac{3a}{2} < x < \frac{5a}{2}$ Thus, $|x+a| < \frac{5a}{2}$.

Also, $\left(\frac{a}{2}\right)^2 < x^2 < \left(\frac{3a}{2}\right)^2$. So, $|x^2| > \frac{a^2}{4}$. Thus $\frac{1}{|x^2|} < \frac{4}{a^2}$

Thus if $|x-a| < \frac{a}{2}$, then

$$\left|\frac{1}{x^2} - \frac{1}{a^2}\right| = \frac{|x-a||x+a|}{|x^2|\,|a^2|} < \left(|x-a|\right)\left(\frac{5a}{2}\right)\left(\frac{4}{a^2}\right)\left(\frac{1}{a^2}\right)$$

So if $|x-a| < \frac{a}{2}$, we have $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| < |x-a| \cdot \frac{10}{a^3}$

Let $\delta = \min\left\{\frac{a}{2}, \frac{\varepsilon}{\left(\frac{10}{a^3}\right)}\right\}$, Therefore, if $|x-a| < \delta$,

then $\left|\frac{1}{x^2} - \frac{1}{a^2}\right| \underset{\uparrow\ |x-a| < \frac{a}{2}}{<} |x-a|\frac{10}{a^3} \underset{\uparrow\ |x-a| < \frac{\varepsilon}{\left(\frac{10}{a^3}\right)}}{<} \frac{\varepsilon}{\left(\frac{10}{a^3}\right)} \cdot \left(\frac{10}{a^3}\right) = \varepsilon$ $\boxtimes$