Math 4570 - Homework # 6

Inner Product Spaces

We begin with some definitions that will be used in this homework assignment.

Definition: Let V be a vector space and W_1 and W_2 be subspaces of V. We say that V is the **direct sum** of W_1 and W_2 and write $V = W_1 \oplus W_2$ if

$$V = W_1 + W_2 = \{w_1 + w_2 \mid w_1 \in W_1, w_2 \in W_2\}$$

and $W_1 \cap W_2 = \{\mathbf{0}\}$.

Definition: Let V be an inner product space over $F = \mathbb{R}$ or $F = \mathbb{C}$. Let W be a subspace of V. Define

$$W^{\perp} = \{v \in V \mid \langle v, w \rangle = 0 \text{ for all } w \in W\}$$

That is, $W^{\perp}$ consists of the vectors from V that are orthogonal to every vector in W.

Definition: Let $A \in M_{n,n}(\mathbb{C})$. The **conjugate transpose** of A is defined to be the matrix A^* where the i,j-th entry of A^* is $(A^*)_{i,j} = \overline{A_{j,i}}$. That is, we transpose A and then conjugate each entry to get A^*. The **trace** of A is the sum of the diagonal elements of A, that is $tr(A) = \sum_{i=1}^{n} A_{i,i}$.

<u>The homework problems begin here.</u>

1. Let $z, w \in \mathbb{C}$. Prove that:

   (a) $\overline{\overline{z}} = z$

   (b) $\overline{z + w} = \overline{z} + \overline{w}$

   (c) $\overline{zw} = \overline{z} \cdot \overline{w}$

   (d) $\overline{\left(\frac{z}{w}\right)} = \frac{\overline{z}}{\overline{w}}$ if $w \neq 0$

   (e) $|z|^2 = z\overline{z}$ where $|a + bi| = \sqrt{a^2 + b^2}$

   (f) $z\overline{z} \in \mathbb{R}$ with $z\overline{z} \geq 0$. Furthermore, $z\overline{z} = 0$ iff $z = 0$.

2. (a) Verify that $\beta = \left\{ \left(\frac{1}{\sqrt{2}}, -\frac{1}{\sqrt{2}} \right), \left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}} \right) \right\}$ is an orthonomal basis for $\mathbb{R}^2$. Then express the vector $w = (3, 7)$ in terms of the elements of β.

   (b) Verify that $\beta = \left\{ \left(\frac{2}{3}, -\frac{2}{3}, \frac{1}{3} \right), \left(\frac{2}{3}, \frac{1}{3}, -\frac{2}{3} \right), \left(\frac{1}{3}, \frac{2}{3}, \frac{2}{3} \right) \right\}$ is an orthonomal basis for $\mathbb{R}^3$. Then express the vector $w = (-1, 0, 2)$ in terms of the elements of β.

3. (a) Transform the following basis of $\mathbb{R}^3$ into an orthonormal basis using the Gram-Schmidt process:

$$\beta = \{(1, 1, 1), (-1, 1, 0), (1, 2, 1)\}$$

   (b) Transform the following basis of $\mathbb{R}^3$ into an orthonormal basis using the Gram-Schmidt process:

$$\beta = \{(1, 0, 0), (3, 7, -2), (0, 4, 1)\}$$

   (c) Transform the following basis of $\mathbb{R}^4$ into an orthonormal basis using the Gram-Schmidt process:

$$\beta = \{(0, 2, 1, 0), (1, -1, 0, 0), (1, 2, 0, -1), (1, 0, 0, 1)\}$$

4. Let V be an inner product space over $F = \mathbb{R}$ or $F = \mathbb{C}$. Then for all $x, y, z \in V$ and $c \in F$ we have

   (a) $\langle x, y + z \rangle = \langle x, y \rangle + \langle x, z \rangle$

   (b) $\langle x, cy \rangle = \overline{c} \langle x, y \rangle$

   (c) $\langle x, x \rangle = 0$ iff $x = 0$

5. Let V be an inner product space over $F = \mathbb{R}$ or $F = \mathbb{C}$. Then for all $x, y \in V$ and $c \in F$ we have

   (a) $\|x\|^2 = \langle x, x \rangle$

   (b) $\|cx\| = |c| \cdot \|x\|$

   (c) $\|x\| \geq 0$. Furthermore, $\|x\| = 0$ iff $x = 0$.

(d) (Pythagorean Theorem) If $\{v_1, v_2, \ldots, v_n\}$ is an orthogonal set in V and $a_1, a_2, \ldots, a_n \in F$ then

$$\left\| \sum_{i=1}^{n} a_i v_i \right\|^2 = \sum_{i=1}^{n} |a_i| \cdot \|v_i\|^2.$$

In particular, setting $a_i = 1$ for all i then we get

$$\left\| \sum_{i=1}^{n} v_i \right\|^2 = \sum_{i=1}^{n} \cdot \|v_i\|^2.$$

6. Let V be an inner product space over $F = \mathbb{R}$ or $F = \mathbb{C}$.

   (a) If $v \in V$, then $\langle \mathbf{0}, v \rangle = \langle v, \mathbf{0} \rangle = 0$

   (b) If $S = \{v_1, v_2, \ldots, v_n\}$ is an orthogonal set of vectors from V, then S is a linearly independent set.

7. Let $V = \mathbb{R}^3$ and $F = \mathbb{R}$. Let $W = \text{span}(\{(1, 0, 0)\})$. Calculate $W^\perp$.

8. Let V be an inner product space over $F = \mathbb{R}$ or $F = \mathbb{C}$. Let W be a subspace of V. Prove:

   (a) $W^\perp$ is a subspace of V.

   (b) $\{\mathbf{0}\}^\perp = \mathbf{V}$

   (c) $V^\perp = \{\mathbf{0}\}$

   (d) If $W_1 \subseteq W_2$, then $W_2^\perp \subset W_1^\perp$.

9. Let V be a vector space and W_1 and W_2 be subspaces of V. Prove that $V = W_1 \oplus W_2$ if and only if every vector $x \in V$ can be expressed uniquely in the form $x = w_1 + w_2$ where $w_1 \in W_1$ and $w_2 \in W_2$.

10. Let V be a finite-dimensional inner product space over $F = \mathbb{R}$ or $F = \mathbb{C}$. Let W be a subspace of V. Prove that $V = W \oplus W^\perp$.