Math 4570 - Homework # 3

Linear Transformations

1. Let V and W be vector spaces over a field F. Let $\mathbf{0_V}$ and $\mathbf{0_W}$ be the zero vectors of V and W respectively. Let $T : V \to W$ be a function. Prove the following.

   (a) If T is a linear transformation, then $T(\mathbf{0_V}) = \mathbf{0_W}$.

   (b) T is a linear transformation if and only if
   $$T(\alpha x + \beta y) = \alpha T(x) + \beta T(y)$$
   for all $x, y \in V$ and $\alpha, \beta \in F$.

   (c) T is a linear transformation if and only if
   $$T(\sum_{i=1}^{n} \alpha_i x_i) = \sum_{i=1}^{n} \alpha_i T(x_i)$$
   for all $x_1, \ldots, x_n \in V$ and $\alpha_1, \ldots, \alpha_n \in F$.

2. Verify whether or not $T : V \to W$ is a linear transformation. If T is a linear transformation then: (i) compute the nullspace of T, (ii) compute the range of T, (iii) compute the nullity of T, (iv) compute the rank of T, (v) determine if T one-to-one, and (vi) determine if T is onto.

   (a) $T : \mathbb{R}^3 \to \mathbb{R}^2$ given by $T(a, b, c) = (a - b, 2c)$

   (b) $T : \mathbb{R}^2 \to \mathbb{R}^2$ given by $T(a, b) = (a - b, b^2)$

   (c) $T : M_{2,3}(\mathbb{R}) \to M_{2,2}(\mathbb{R})$ given by
   $$T\begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} = \begin{pmatrix} 2a - b & c + 2d \\ 0 & 0 \end{pmatrix}$$

   (d) $T : P_2(\mathbb{R}) \to P_3(\mathbb{R})$ given by $T(a + bx + cx^2) = a + bx^3$

   (e) $T : P_2(\mathbb{R}) \to P_2(\mathbb{R})$ given by $T(a + bx + cx^2) = (1 + a) + (1 + b)x + (1 + c)x^2$

3. Let a and b be real numbers where $a < b$. Let $C(\mathbb{R})$ be the vector space of continuous functions on the real line as in HW # 1. Let $T : C(\mathbb{R}) \to \mathbb{R}$ given by

$$T(f) = \int_a^b f(t)dt$$

Verify whether or not T is linear.

4. Let F be a field. Recall that if $A \in M_{m,n}(F)$ then we can make a linear transformation $L_A : F^n \to F^m$ where $L_A(x) = Ax$ is left-sided matrix multiplication. In each problem, calculate $L_A(x)$ for the given A and x.

   (a) $F = \mathbb{R}$, $L_A : \mathbb{R}^2 \to \mathbb{R}^2$, $A = \begin{pmatrix} 1 & \pi \\ \frac{1}{2} & -10 \end{pmatrix}$, $x = \begin{pmatrix} 17 \\ -5 \end{pmatrix}$

   (b) $F = \mathbb{C}$, $L_A : \mathbb{C}^3 \to \mathbb{C}^2$, $A = \begin{pmatrix} -i & 1 & 0 \\ 1+i & 0 & -1 \end{pmatrix}$, $x = \begin{pmatrix} -2i \\ 4 \\ 1.57 \end{pmatrix}$

5. Let V and W be vector spaces over a field F. Let $T : V \to W$ be a linear transformation. Let $v_1, \ldots, v_n \in V$ such that $\mathrm{span}(\{v_1, \ldots, v_n\}) = V$, then $\mathrm{span}(\{T(v_1), \ldots, T(v_n)\}) = R(T)$.

6. Let V and W be vector spaces over a field F. Let $T : V \to W$ be a linear transformation. Let $\mathbf{0_V}$ and $\mathbf{0_W}$ be the zero vectors of V and W respectively.

   (a) Prove that T is one-to-one if and only if $N(T) = \{\mathbf{0_V}\}$.

   (b) Suppose that V and W are both finite-dimensional and $\dim(V) = \dim(W)$. Prove that T is one-to-one if and only if T is onto.

   (c) Suppose that V and W are both finite-dimensional. Prove that if T is one-to-one and onto then $\dim(V) = \dim(W)$.

7. Let V and W be finite dimensional vector spaces and let $T : V \to W$ be a linear transformation.

   (a) If $\dim(V) < \dim(W)$, then T is not onto.

   (b) If $\dim(V) > \dim(W)$, then T is not one-to-one.