Math 4570 - Homework # 2

Spanning sets, Linear Independence, Bases, Dimension

Recall from HW 1: Let V be a vector space over a field F. Let W_1 and W_2 be subspaces of V. Define the **sum** of W_1 and W_2 to be the set

$$W_1 + W_2 = \{x + y \mid x \in W_1 \text{ and } y \in W_2\}$$

1. Let $V = M_{2,2}(\mathbb{R})$ and

$$W_1 = \left\{ \begin{pmatrix} a & b \\ c & a \end{pmatrix} \mid a, b, c \in \mathbb{R} \right\}$$

and

$$W_2 = \left\{ \begin{pmatrix} 0 & a \\ -a & b \end{pmatrix} \mid a, b \in \mathbb{R} \right\}$$

   (a) Prove that W_1 and W_2 are subspaces of $M_{2,2}(\mathbb{R})$.

   (b) Find the dimensions of W_1, W_2, $W_1 \cap W_2$ and $W_1 + W_2$.

2. Let V be a vector space over a field F. Let $v_1, v_2, \cdots, v_n$ be vectors in V. Prove that if one of the v_i is the zero vector, then the vectors $v_1, v_2, \ldots, v_n$ are linearly dependent.

3. Let F be either $\mathbb{R}$ or $\mathbb{C}$. Prove that $P_n(F)$ has dimension $n + 1$.

4. Let $P(\mathbb{R})$ denote the set of all polynomials with coefficients from $\mathbb{R}$. You may assume that $P(\mathbb{R})$ is a vector space over $\mathbb{R}$. Show that $P(\mathbb{R})$ is not finite dimensional.

5. Let V be a vector space over a field F. Let $x, y \in V$. Then $\{x, y\}$ is a linearly dependent set if and only if x or y is a multiple of the other.

6. Let V be a vector space over a field F. Let $x \in V$ with $x \neq \mathbf{0}$. Then $\{x\}$ is a linearly independent set.

7. Let V be a vector space over a field F.

(a) Let S be a finite set of linearly independent vectors from V and let $v \in V$ where $v \notin S$. Then $S \cup \{v\}$ is linearly dependent if and only if $v \in \text{span}(S)$.

(b) Suppose that $V \neq \{\mathbf{0}\}$ is spanned by some finite set S. Prove that some subset of S is a basis for V. Thus V is finite-dimensional.

8. Let W_1 and W_2 be subspaces of a vector space V. Suppose that $\dim(W_1) = m$ and $\dim(W_2) = n$ and $m \leq n$.

   (a) Prove that $\dim(W_1 \cap W_2) \leq m = \min(\dim(W_1), \dim(W_2))$

   (b) Prove that $\dim(W_1 + W_2) \leq m + n$

9. (This problem shows how to extend a basis from a subspace of a finite-dimensional vector space to the entire space.) Let V be a finite-dimensional vector space of dimension $n \neq 0$ over a field F. Let W be a subspace of V with $W \neq \{\mathbf{0}\}$. In class we showed that W must be finite-dimensional and hence have a basis $\beta = \{w_1, w_2, \ldots, w_k\}$ with $1 \leq k \leq n$. Prove that there exist vectors $v_{k+1}, \ldots, v_n$ from $V \setminus W$ such that $\beta' = \{w_1, w_2, \ldots, w_k, v_{k+1}, \ldots, v_n\}$ is a basis for V.