Math 456

Homework # 1 - Rings and Fields

1. Are the following sets R rings with the given operations? Show why. For each R that is a ring, also answer the following questions: (a) Is R commutative? (b) Does R have a multiplicative identity? (c) If R has a multiplicative identity, find all of the units of R. (d) Is R a field?

(a) $R = \mathbb{Z}^+ = \{1, 2, 3, 4, \ldots\}$ with the usual $+$ and $\cdot$

(b) The Gaussian integers $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$ with the usual $+$ and $\cdot$

(c) The imaginary axis $R = \{ix \mid x \in \mathbb{R}\}$ with the usual $+$ and $\cdot$

(d) The quadratic number field $R = \mathbb{Q}(\sqrt{2}) = \{a + b\sqrt{2} \mid a, b \in \mathbb{Q}\}$ with the usual $+$ and $\cdot$

2. Which of the following are subrings of $M_2(\mathbb{R})$?

(a) $R_1 = \left\{ \begin{pmatrix} a & 0 \\ 0 & b \end{pmatrix} \;\middle|\; a, b \in \mathbb{Z} \right\}$

(b) $R_2 = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \;\middle|\; a, b, c, d \in \mathbb{R} \text{ and } ad - bc = 1 \right\}$

3. Find the units in the following rings.

(a) $\mathbb{Z} \times \mathbb{Z}$

(b) $\mathbb{Z}_2 \times \mathbb{Z}_3$

(c) $R = \mathbb{Z}[i] = \{a + bi \mid a, b \in \mathbb{Z}\}$

4. Let R be a ring with multiplicative identity. Prove that the multiplicative identity is unique.

5. Let R be a ring with multiplicative identity. Let x be a unit in R. Prove that there is a unique multiplicative inverse for x.

6. Let R be a ring and a be a fixed element of R. Let

$$I_a = \{x \in R \mid ax = 0\}.$$

Prove that I_a is a subring of R.

7. Let $n \in \mathbb{Z}$ with $n \geq 0$. Prove that

$$n\mathbb{Z} = \{nx \mid x \in \mathbb{Z}\}$$

is a subring of $\mathbb{Z}$.

8. Let R be a commutative ring with identity $1 \neq 0$. Let $R^\times$ be the set of units of R. Prove that $R^\times$ is a group under multiplication.

9. Let R and S be subrings of a ring T. Prove that $R \cap S$ is a subring of T.