HW #6

(1)

(a) $\mathbb{Z}_{12}$ is abelian so the left and right cosets are equal.

$H = \{\bar{0}, \bar{4}, \bar{8}\}$

$\bar{1} + H = \{\bar{1}, \bar{5}, \bar{9}\} = H + \bar{1}$

$\bar{2} + H = \{\bar{2}, \bar{6}, \bar{10}\} = H + \bar{2}$

$\bar{3} + H = \{\bar{3}, \bar{7}, \bar{11}\} = H + \bar{3}$

(b) $\mathbb{Z}$ is abelian so the left and right cosets are equal.

$0 + 4\mathbb{Z} = \{\ldots, -12, -8, -4, 0, 4, 8, 12, \ldots\} = 4\mathbb{Z} + 0$

$1 + 4\mathbb{Z} = \{\ldots, -11, -7, -3, 1, 5, 9, 13, \ldots\} = 4\mathbb{Z} + 1$

$2 + 4\mathbb{Z} = \{\ldots, -10, -6, -2, 2, 6, 10, 14, \ldots\} = 4\mathbb{Z} + 2$

$3 + 4\mathbb{Z} = \{\ldots, -9, -5, -1, 3, 7, 11, 15, \ldots\} = 4\mathbb{Z} + 3$

(c) $S_3 = \{\bar{\imath}, (1,2), (2,3), (1,3), (1,2,3), (1,3,2)\}$

$H = \{\bar{\imath}, (1,2)\}$

$2,3)\ H = \{(2,3)\bar{\imath}, (2,3)(1,2)\} = \{(2,3), (1,3,2)\}$

$,3)\ H = \{(1,3)\bar{\imath}, (1,3)(1,2)\} = \{(1,3), (1,2,3)\}$

$H = \{\bar{\imath}, (1,2)\}$

$H(2,3) = \{\bar{\imath}(2,3), (1,2)(2,3)\} = \{(2,3), (1,2,3)\}$

$H(1,3) = \{\bar{\imath}(1,3), (1,2)(1,3)\} = \{(1,3), (1,3,2)\}$

Left and right cosets are not equal

More space for problems 7 and 8

(d) $H = \{ i, (1,2,3), (1,3,2) \}$

$(1,2)H = \{ (1,2)i, (1,2)(1,2,3), (1,2)(1,3,2) \}$

$= \{ (1,2), (2,3), (1,3) \}$

$H = \{ i, (1,2,3), (1,3,2) \}$

$H(1,2) = \{ i(1,2), (1,2,3)(1,2), (1,3,2)(1,2) \}$

$= \{ (1,2), (1,3), (2,3) \}.$

So, the left and right cosets are equal.

(e) $D_8 = \{ 1, r, r^2, r^3, s, sr, sr^2, sr^3 \}$

$H = \langle r \rangle = \{ 1, r, r^2, r^3 \}$

$sH = \{ s, sr, sr^2, sr^3 \}$

$H = \{ 1, r, r^2, r^3 \}$

$Hs = \{ 1s, rs, r^2s, r^3s \} = \{ s, sr^3, sr^2, sr \}$

So, the left and right cosets are equal.

(f) $D_8 = \{ 1, r, r^2, r^3, s, sr, sr^2, sr^3 \}$

$H = \{ 1, s \}$

$rH = \{ r, rs \} = \{ r, sr^3 \}$

$r^2H = \{ r^2, r^2s \} = \{ r^2, sr^2 \}$

$r^3H = \{ r^3, r^3s \} = \{ r^3, sr \}$

Not equal.

$H = \{ 1, s \}$

$Hr = \{ r, sr \}$

$Hr^2 = \{ r^2, sr^2 \}$

$Hr^3 = \{ r^3, sr^3 \}$

(2) Suppose that $aH = bH$.

Let $x \in Ha^{-1}$. Then $x = ha^{-1}$ for some $h \in H$.

Then $x^{-1} = (ha^{-1})^{-1} = ah^{-1}$. So, $x^{-1} \in aH$.

Since $aH = bH$, we have that $x^{-1} \in bH$.

So, $x^{-1} = bh_1$ for some $h_1 \in H$.

Thus, $x = (bh_1)^{-1} = h_1^{-1} b^{-1} \in Hb^{-1}$.

Therefore $Ha^{-1} \subseteq Hb^{-1}$.

A similar argument shows that $Hb^{-1} \subseteq Ha^{-1}$.

Try to write it out.

This will show that $Ha^{-1} = Hb^{-1}$.

More space for problems 7 and 8

(3) Let H be a subgroup of G with $H \neq G$.
By Lagrange's Thm, $|H|$ divides $|G| = pq$.
Since p and q are primes, $|H| = 1, p, q,$ or pq.
Since $H \neq G$, $|H| = 1, p,$ or q.
If $|H| = p$ or $|H| = q$, then by in-class proof, H is cyclic.
If $|H| = 1$, then $H = \langle e \rangle$. So, H is cyclic.

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(4) Suppose that x has order k.
Then $\langle x \rangle = \{1, x, x^2, \ldots, x^{k-1}\}$ and $|\langle x \rangle| = k$.
By Lagranges theorem, $k = |\langle x \rangle|$ divides $|G| = n$. Thus, $kl = n$ for some $l \in \mathbb{Z}$.
So, $x^n = x^{kl} = (x^k)^l = e^l = e$.

More space for problems 7 and 8