Hw #4

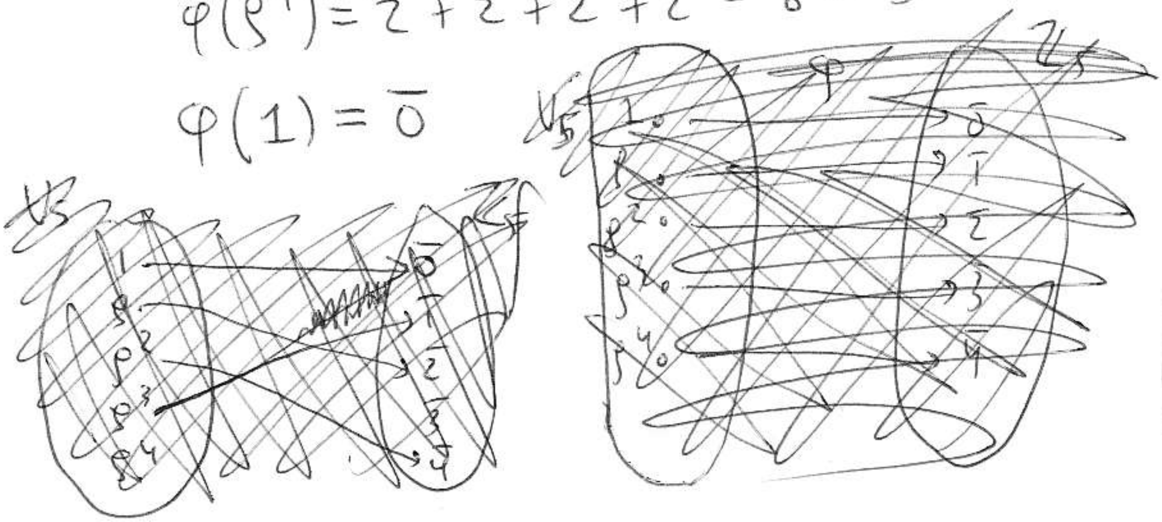
Let $\rho = e^{2\pi i/5}$. Since φ is a hom, $\varphi(xy) = \varphi(x) + \varphi(y)$ for all $x, y \in U_5$.

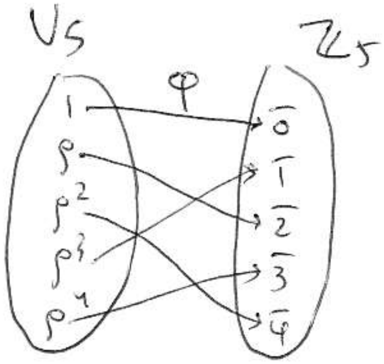
① $\varphi(\rho^2) = \varphi(\rho) + \varphi(\rho) = \bar{2} + \bar{2} = \bar{4}$

$\varphi(\rho^3) = \varphi(\rho) + \varphi(\rho) + \varphi(\rho) = \bar{2} + \bar{2} + \bar{2} = \bar{6} = \bar{1}$

$\varphi(\rho^4) = \bar{2} + \bar{2} + \bar{2} + \bar{2} = \bar{8} = \bar{3}$

$\varphi(1) = \bar{0}$





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② Let $\rho = e^{2\pi i/8} = e^{\pi i/4}$. Suppose φ is a hom from U_8 to $\mathbb{Z}_8$ with $\varphi(\rho) = \bar{2}$.

Then,

$$\left.\begin{aligned}
\varphi(1) &= \bar{0} \\
\varphi(\rho) &= \bar{2} \\
\varphi(\rho^2) &= \varphi(\rho) + \varphi(\rho) = \bar{2} + \bar{2} = \bar{4} \\
\varphi(\rho^3) &= \bar{2} + \bar{2} + \bar{2} = \bar{6} \\
\varphi(\rho^4) &= \bar{0} \\
\varphi(\rho^5) &= \bar{2} \\
\varphi(\rho^6) &= \bar{4} \\
\varphi(\rho^7) &= \bar{6}
\end{aligned}\right\}$$

So, φ is not onto $\mathbb{Z}_8$.
Thus, φ is not an isomorphism (it isn't 1-1 either)

③ Let $\varphi : \mathbb{Z} \to \mathbb{Z}$ be a homomorphism.
Suppose that $\varphi(1) = k$ where $k \in \mathbb{Z}$.
Let $n > 0$. Then,

$$\varphi(n) = \varphi(\underbrace{1+1+\dots+1}_{n \text{ times}}) = \underbrace{\varphi(1)+\varphi(1)+\dots+\varphi(1)}_{n \text{ times}} = nk.$$

since φ is a hom.

And

$$\varphi(-n) = \varphi(\underbrace{(-1)+(-1)+\dots+(-1)}_{n \text{ times}}) = \underbrace{\varphi(-1)+\varphi(-1)+\dots+\varphi(-1)}_{n \text{ times}}$$

$$= \underbrace{-k-k-\dots-k}_{\text{times}}$$

$$= (-n)k.$$

Thus, $\varphi(x) = xk$.

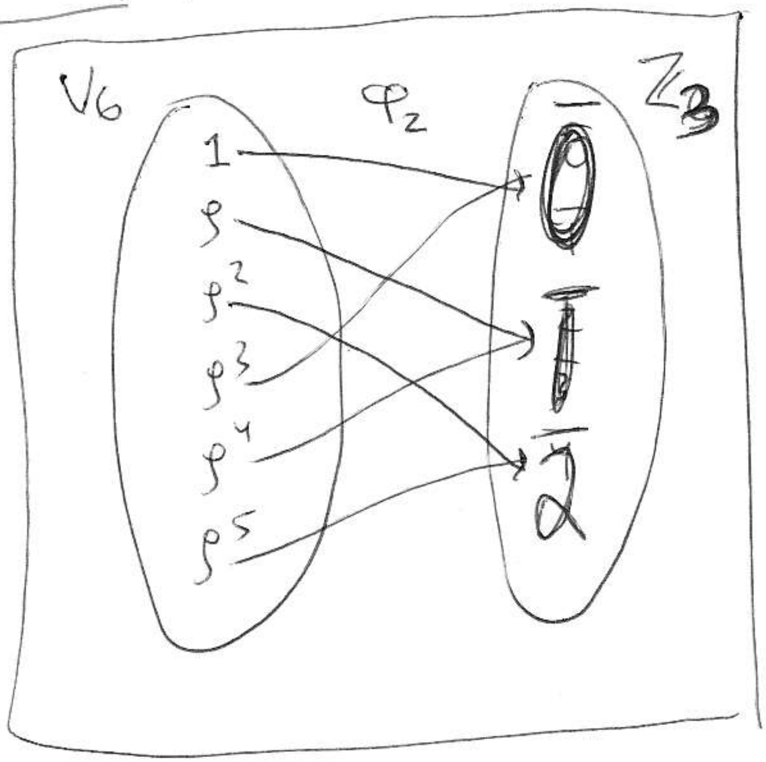
[Note that any function $\varphi : \mathbb{Z} \to \mathbb{Z}$ of the form $\varphi(x) = xk$ where $k \in \mathbb{Z}$ is a homomorphism since $\varphi(x+y) = (x+y)k = xk + yk = \varphi(x) + \varphi(y)$ for all $x, y \in \mathbb{Z}$.]

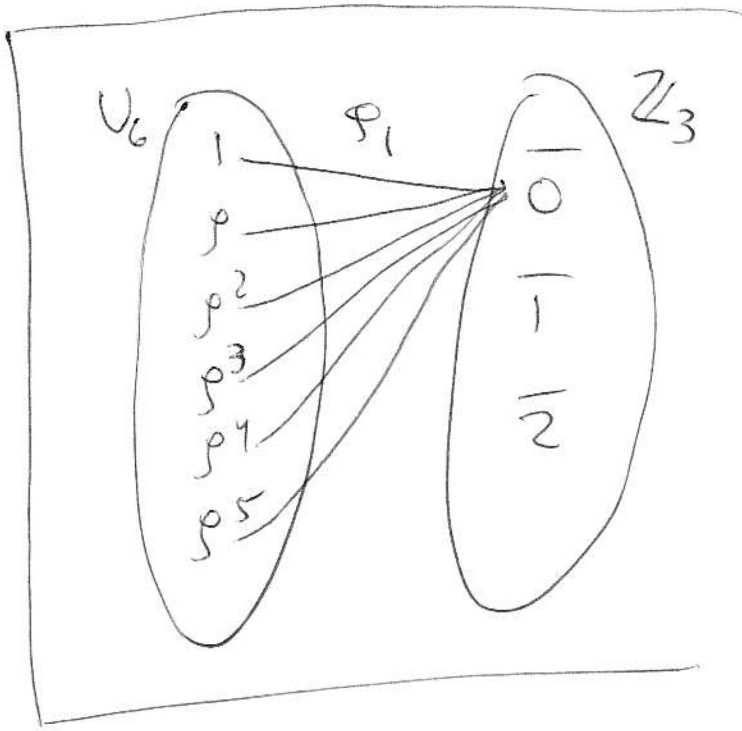
Hence all the homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$ are of the form $\varphi(x) = kx$ where $k \in \mathbb{Z}$.
φ is an isomorphism iff $k = 1$ or $k = -1$.
[Try and prove it.]

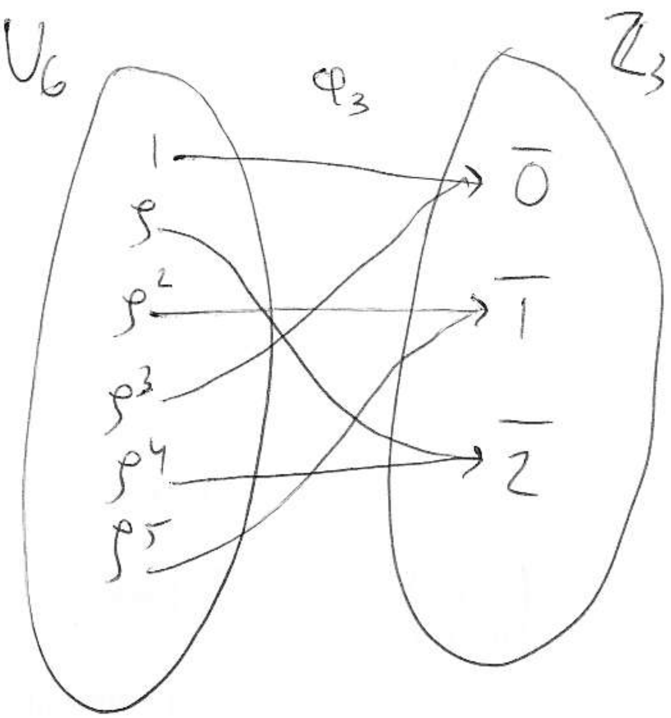
④ $U_6 = \{1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$ where $\rho = e^{2\pi i/6} = e^{\pi i/3}$.

There are 3 possible homomorphisms, depending on where ρ goes. ρ must go to an element of $\mathbb{Z}_3$ of order dividing $6 = \text{order}(\rho)$. All the elements of $\mathbb{Z}_3$ satisfy this condition.

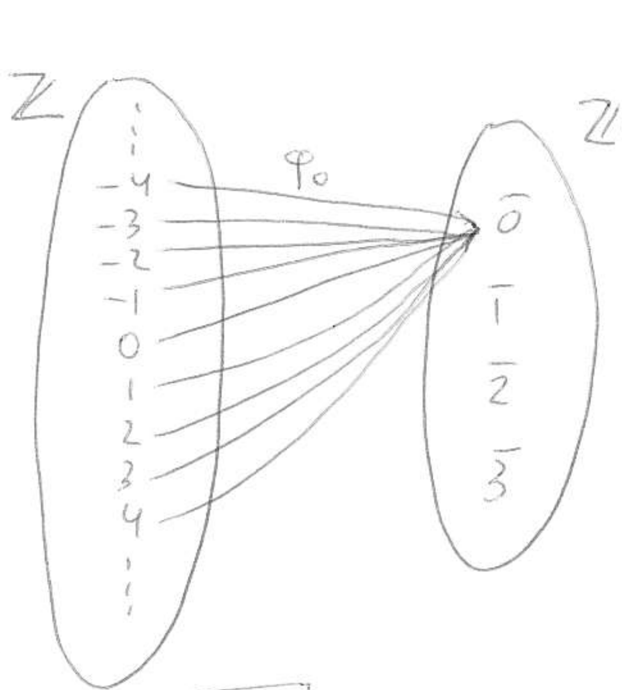
Possible homs: $\varphi_1(\rho) = \bar{0}$ or $\varphi_2(\rho) = \bar{1}$ or $\varphi_3(\rho) = \bar{2}$



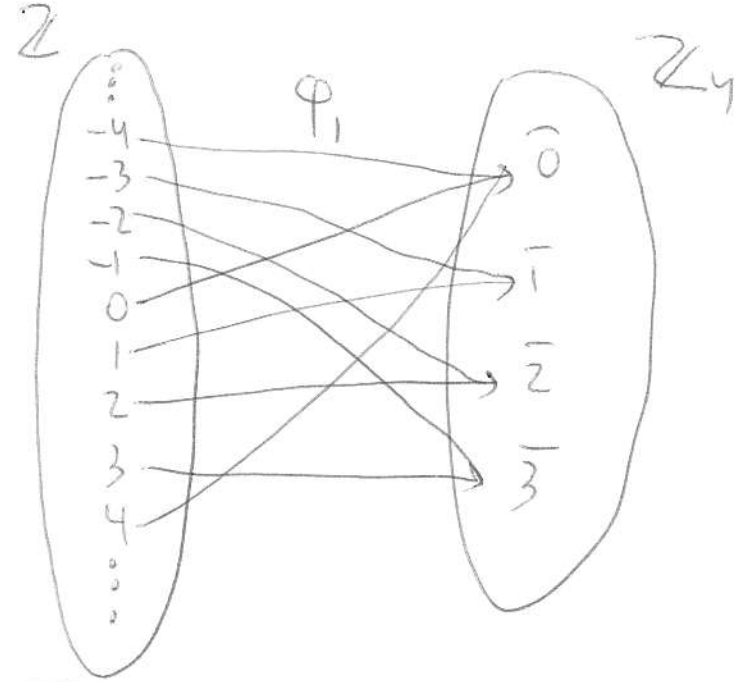




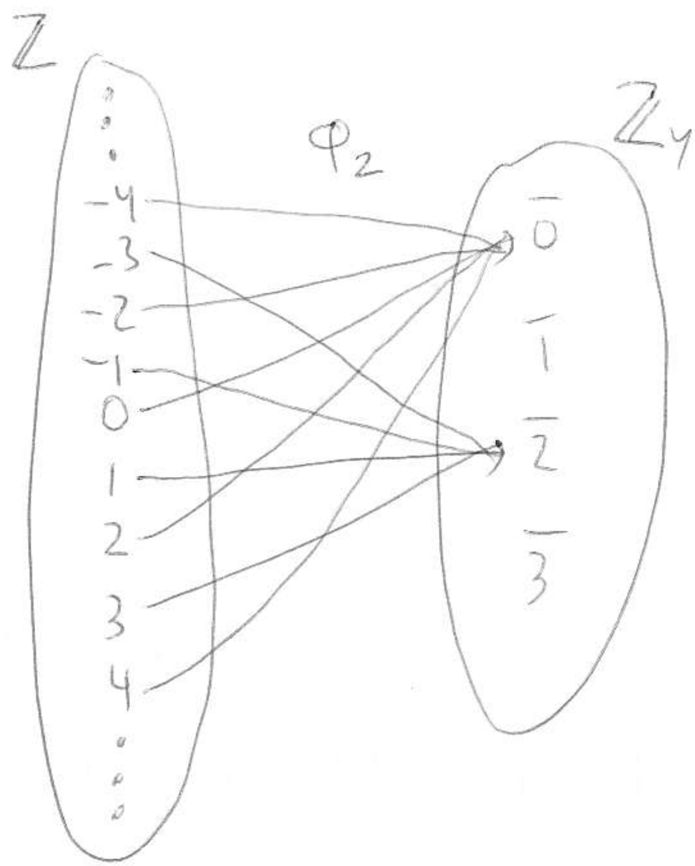
(5) A homomorphism $\varphi : \mathbb{Z} \to \mathbb{Z}_4$ is determined by $\varphi(1)$.



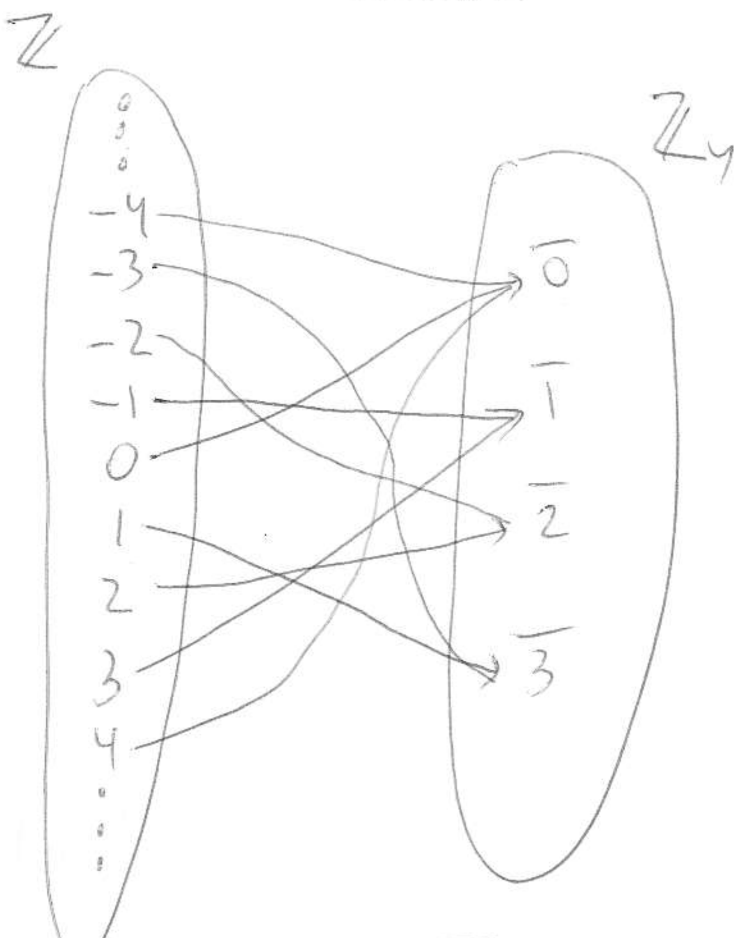
$\boxed{\varphi_0(1) = \bar{0}}$



$\boxed{\varphi_1(1) = \bar{1}}$

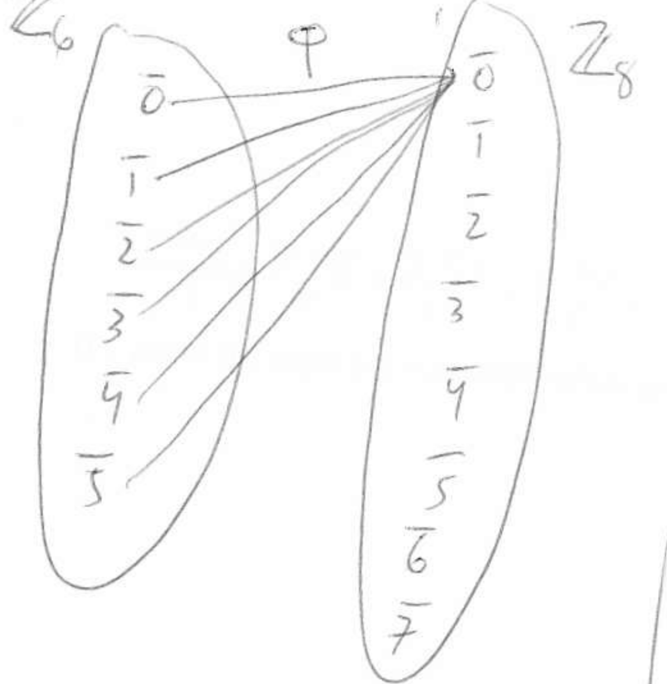


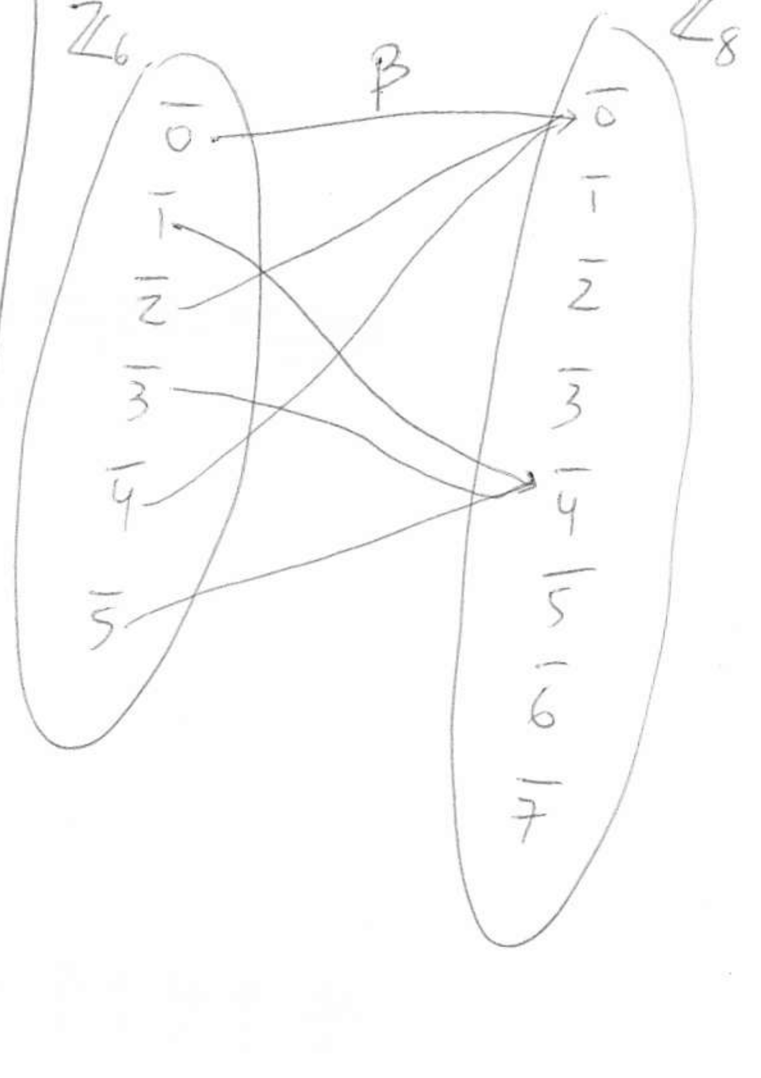
$\boxed{\varphi_2(1) = \bar{2}}$



$\boxed{\varphi_3(1) = \bar{3}}$

⑥ A homomorphism $\varphi : \mathbb{Z}_6 \to \mathbb{Z}_8$ is determined by $\varphi(\bar{1})$. $\bar{1}$ has order 6 in $\mathbb{Z}_6$. Thus $\varphi(\bar{1})$ must have order dividing 6. So, $\varphi(\bar{1})$ must have order 1, 2, or 3. The only elements of $\mathbb{Z}_8$ of these orders are $\bar{0}$ (which has order 1) and $\bar{4}$ (which has order 2). [No element of $\mathbb{Z}_8$ has order 3.] So we get 2 homomorphisms:





(7) $U_6 = \{1, \rho, \rho^2, \rho^3, \rho^4, \rho^5\}$ where $\rho = e^{2\pi i/6} = e^{\pi i/3}$

$\langle 1 \rangle = \{1\}$

$\langle \rho \rangle = U_6 = \langle \rho^5 \rangle$

$\langle \rho^2 \rangle = \{1, \rho^2, \rho^4\} = \langle \rho^4 \rangle$

$\langle \rho^3 \rangle = \{1, \rho^3\}$

So, the subgroups of U_6 are $\{1\}$, $\{1, \rho^3\}$, $\{1, \rho^2, \rho^4\}$ and U_6.

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(8) $\langle \bar{0} \rangle = \{\bar{0}\}$

$\langle \bar{1} \rangle = \mathbb{Z}_8 = \langle \bar{3} \rangle = \langle \bar{5} \rangle = \langle \bar{7} \rangle$

$\langle \bar{2} \rangle = \{\bar{0}, \bar{2}, \bar{4}, \bar{6}\} = \langle \bar{6} \rangle$

$\langle \bar{4} \rangle = \{\bar{0}, \bar{4}\}$

The subgroups of $\mathbb{Z}_8$ are
$\{\bar{0}\}$, $\{\bar{0}, \bar{4}\}$, $\{\bar{0}, \bar{2}, \bar{4}, \bar{6}\}$, and $\mathbb{Z}_8$

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Yes. $\mathbb{Z}_2 = \{\overline{0}, \overline{1}\}$ has only one generator. It is $\overline{1}$.

Yes. $\mathbb{Z}_3 = \{\overline{0}, \overline{1}, \overline{2}\}$ has exactly 2 generators: $\overline{1}$ and $\overline{2}$.

$\mathbb{Z}$ has exactly two generators also: 1 and -1.

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10 Let $\rho = e^{2\pi i/6} = e^{\pi i/3}$.

Then $\langle \rho \rangle = \langle \rho^5 \rangle = U_6$.

The only generators are ρ and ρ^5.

$\boxed{\text{See } \#7}$

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11 $\overline{1}, \overline{3}, \overline{5}$ and $\overline{7}$. See # 8.

(12) Let $x \in G$. Then

$$\langle x \rangle = \{\dots, x^{-3}, x^{-2}, x^{-1}, e, x, x^2, x^3, \dots\}$$

$$= \{\dots, (x^{-1})^3, (x^{-1})^2, x^{-1}, e, x, x^2, x^3, \dots\}$$

$$= \{\dots, x^3, x^2, x, e, x^{-1}, (x^{-1})^2, (x^{-1})^3, \dots\}$$

$$= \{\dots, ((x^{-1})^{-1})^3, ((x^{-1})^{-1})^2, ((x^{-1})^{-1})^1, e, x^{-1}, (x^{-1})^2, (x^{-1})^3, \dots\}$$

$$= \langle x^{-1} \rangle.$$

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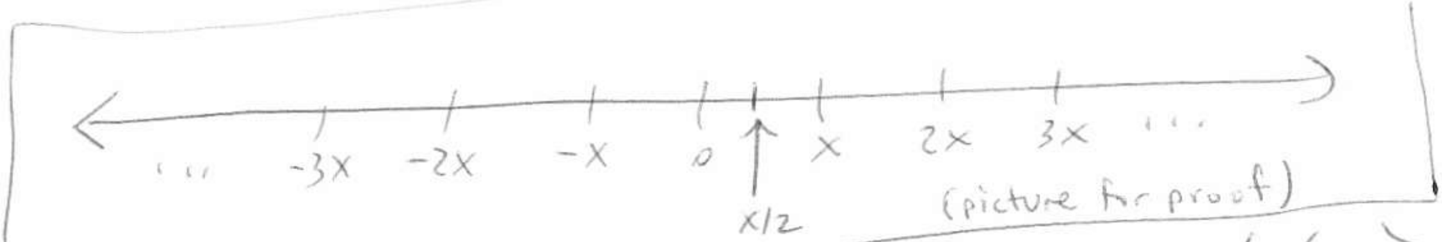
(13) Suppose that $x \in \mathbb{Q}$, and $x \neq 0$.

Then,

$$\langle x \rangle = \{\dots, -3x, -2x, -x, 0, x, 2x, 3x, \dots\}$$



Then $x/2 \in \mathbb{Q}$, but $\frac{x}{2} \notin \langle x \rangle$. So, $\mathbb{Q} \neq \langle x \rangle$.

(14) By #12, $|\langle x \rangle| = |\langle x^{-1} \rangle|$.

In class we showed that the order of x is equal to $|\langle x \rangle|$ and the order of x^{-1} is equal to $|\langle x^{-1} \rangle|$. Hence, $|\langle x \rangle| = |\langle x^{-1} \rangle|$.

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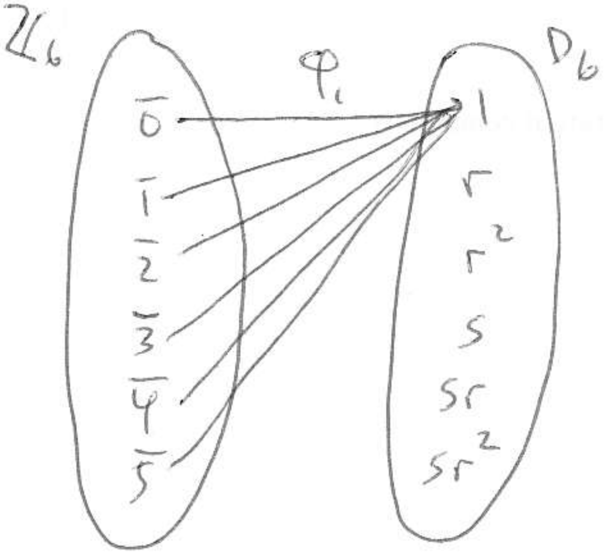
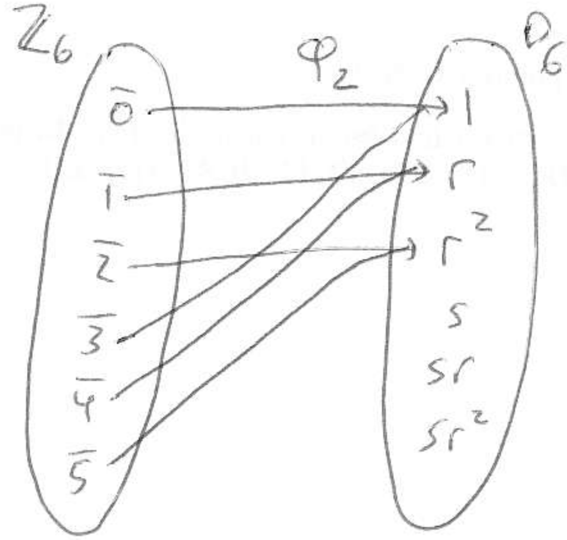
(15) $\mathbb{Z}_6 = \langle \bar{1} \rangle$ and $\bar{1}$ has order 6. So if $\varphi: \mathbb{Z}_6 \to D_6$ is a homomorphism then $\varphi(\bar{1})$ has order dividing 6. Every element of D_6 has order dividing D_6.

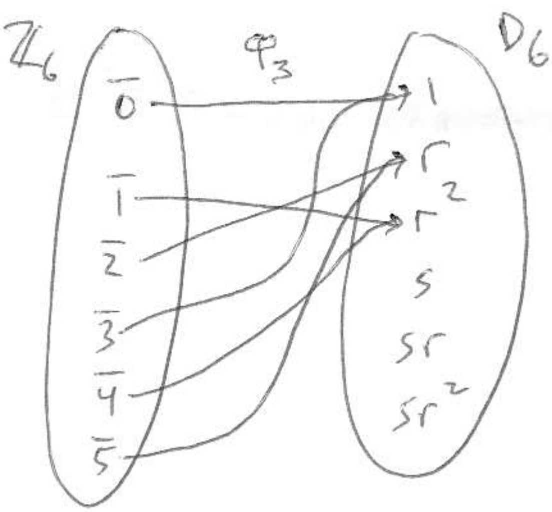
D_6	1	r	r^2	s	sr	sr^2
order	1	3	3	2	2	2

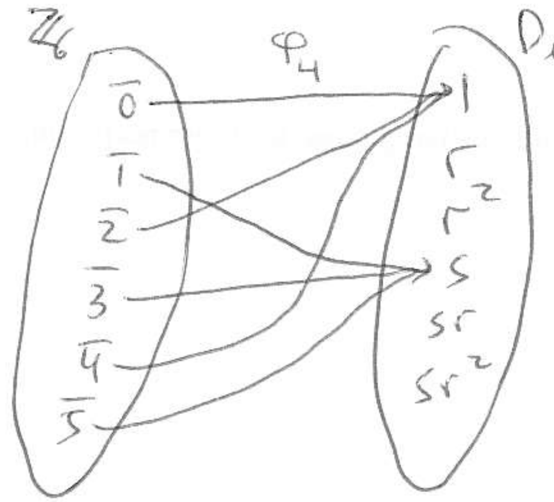
We get the following homomorphisms:

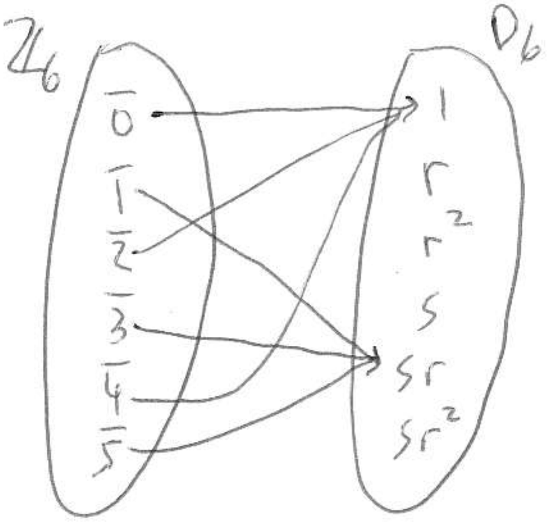
[<u>Recall</u>: One we decide $\varphi(\bar{1}) = x$
then $\varphi(\bar{k}) = \varphi(\underbrace{\bar{1}+\bar{1}+\dots+\bar{1}}_{k \text{ times}}) = \varphi(\bar{1})^k = x^k$]

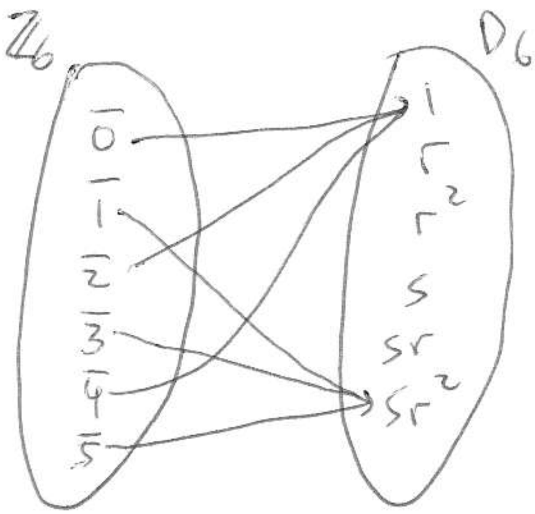

$\mathbb{Z}_6$
φ_1
D_6
$\bar{0}$ $\bar{1}$ $\bar{2}$ $\bar{3}$ $\bar{4}$ $\bar{5}$
1 r r^2 s sr sr^2


$\mathbb{Z}_6$
φ_2
D_6
$\bar{0}$ $\bar{1}$ $\bar{2}$ $\bar{3}$ $\bar{4}$ $\bar{5}$
1 r r^2 s sr sr^2


$\mathbb{Z}_6$
φ_3
D_6
$\bar{0}$ $\bar{1}$ $\bar{2}$ $\bar{3}$ $\bar{4}$ $\bar{5}$
1 r r^2 s sr sr^2


$\mathbb{Z}_6$
φ_4
D_6
$\bar{0}$ $\bar{1}$ $\bar{2}$ $\bar{3}$ $\bar{4}$ $\bar{5}$
1 r r^2 s sr sr^2


$\mathbb{Z}_6$
D_6
$\bar{0}$ $\bar{1}$ $\bar{2}$ $\bar{3}$ $\bar{4}$ $\bar{5}$
1 r r^2 s sr sr^2


$\mathbb{Z}_6$
D_6
$\bar{0}$ $\bar{1}$ $\bar{2}$ $\bar{3}$ $\bar{4}$ $\bar{5}$
1 r r^2 s sr sr^2

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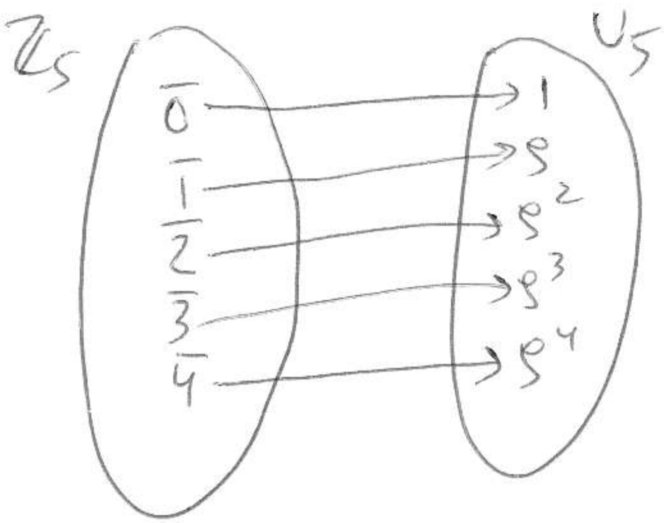
(a) $\mathbb{R}$ is uncountable and $\mathbb{Z}$ is countable, hence no isomorphism exists between them. Or, $\mathbb{R}$ is not cyclic [Mimick the proof I gave to show that $\mathbb{Q}$ is not cyclic.] and $\mathbb{Z}$ is cyclic. So, $\mathbb{R} \not\cong \mathbb{Z}$.

(b) $\mathbb{Z}_5 \cong U_5$



(c) D_8 is not cyclic and $\mathbb{Z}_8$ is cyclic. So, $D_8 \not\cong \mathbb{Z}_8$.

(d) $\mathbb{C}^*$ has 2 elements of order 4, they are i and $-i$. $\mathbb{R}^*$ has no elements of order 4. Hence $\mathbb{C}^* \not\cong \mathbb{R}^*$ by hw #3 problem 5(b).