Math 446 - Homework # 2

1. *For the numbers a and b given below do the following: (i) list the positive divisors of a, (ii) list the positive divisors of b, (iii) list the positive common divisors of a and b, (iv) calculate* $\gcd(a, b)$.

   Solutions:

   (a) $a = 12$ and $b = 24$.

   (i) The divisors of $a = 12$ are 1, 2, 3, 4, 6, and 12. (ii) The divisors of $b = 24$ are 1, 2, 3, 4, 6, 8, 12, and 24. (iii) The common divisors of 12 and 24 are 1, 2, 3, 4, 6 and 12. (iv) Therefore $\gcd(12, 24) = 12$.

   (b) $a = 16$ and $b = 36$

   (i) The divisors of $a = 16$ are 1, 2, 4, 8, and 16. (ii) The divisors of $b = 36$ are 1, 2, 3, 4, 6, 9, 12, 18, and 36. (iii) The common divisors of 16 and 36 are 1, 2, and 4. (iv) Therefore $\gcd(12, 24) = 4$.

   (c) $a = 5$ and $b = 18$

   (i) The divisors of $a = 5$ are 1, and 5. (ii) The divisors of $b = 18$ are 1, 2, 3, 6, 9, and 18. (iii) The common divisor of 5 and 18 are 1. (iv) Therefore $\gcd(5, 18) = 1$.

   (d) $a = 0$ and $b = 3$

   (i) Every non-zero integer k divides $a = 0$ since $k \cdot 0 = 0$. (ii) The divisors of $b = 3$ are 1, and 3. (iii) The common divisor of 0 and 3 are 1 and 3. (iv) Therefore $\gcd(0, 3) = 3$.

2. *Calculate the following:*

   Solutions:

   (a) $\gcd(12, 25, 14)$

   The positive divisors of 12 are 1, 2, 3, 4, 6, and 12. The positive divisors of 25 are 1, 5, and 25. The positive divisors of 14 are 1, 2, 7, and 14. The only positive common divisor of 12, 25, and 14 is the integer 1. Therefore, $\gcd(12, 25, 14) = 1$.

   (b) $\gcd(30, 6, 10)$

   The positive divisors of 30 are 1, 2, 3, 5, 6, 10, 15 and 30. The positive divisors of 6 are 1, 2, 3, and 6. The positive divisors of

10 are 1, 2, 5, and 10. The positive common divisors of 30, 6, and 10 are 1 and 2. Therefore, $\gcd(30, 6, 10) = 2$.

(c) $\gcd(12, 0, 8)$

The positive divisors of 12 are 1, 2, 3, 4, 6, and 12. Every positive integer divides 0. The positive divisors of 8 are 1, 2, 4, and 8. The positive common divisors of 12, 0, and 8 are 1, 2, and 4. Therefore, $\gcd(12, 0, 8) = 4$.

3. *Using the Euclidean algorithm, calculate the greatest common divisor of the following numbers:*

Solutions:

(a) 39 and 17

$$\begin{aligned} 39 &= 2 \cdot 17 + 5 \\ 17 &= 3 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned}$$

The final non-zero remainder is 1. Hence $\gcd(39, 17) = 1$.

(b) 2689 and 4001

$$\begin{aligned} 4001 &= 1 \cdot 2689 + 1312 \\ 2689 &= 2 \cdot 1312 + 65 \\ 1312 &= 20 \cdot 65 + 12 \\ 65 &= 5 \cdot 12 + 5 \\ 12 &= 2 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned}$$

The final non-zero remainder is 1. Hence $\gcd(2689, 4001) = 1$.

(c) 1819 and 3587

$$\begin{aligned}
3587 &= 1 \cdot 1819 + 1768 \\
1819 &= 1 \cdot 1768 + 51 \\
1768 &= 34 \cdot 51 + 34 \\
51 &= 1 \cdot 34 + 17 \\
34 &= 2 \cdot 17
\end{aligned}$$

The final non-zero remainder is 17. Hence $\gcd(3587, 1819) = 17$.

(d) 864 and 468

$$\begin{aligned}
864 &= 1 \cdot 468 + 396 \\
468 &= 1 \cdot 396 + 72 \\
396 &= 5 \cdot 72 + 36 \\
72 &= 2 \cdot 36
\end{aligned}$$

The final non-zero remainder is 36. Hence $\gcd(864, 468) = 36$.

4. *For each problem: First determine if there are any integer solutions. If there are no solutions, explain why not. If there are solutions, then carry out these steps: (a) Use the Euclidean algorithm to find integers x and y that satisfy the equation, (b) give a formula for all the solutions to the equation, and (c) use your formula to find four more solutions to the equation.*

Solutions:

(a) $4001x + 2689y = 1$

We have that $\gcd(4001, 2689) = 1$. Since 1 divides 1, we know that $4001x + 2689y = 1$ has integer solutions. Using the Euclidean algorithm we have that

$$
\begin{aligned}
4001 &= 1 \cdot 2689 + 1312 \\
2689 &= 2 \cdot 1312 + 65 \\
1312 &= 20 \cdot 65 + 12 \\
65 &= 5 \cdot 12 + 5 \\
12 &= 2 \cdot 5 + 2 \\
5 &= 2 \cdot 2 + 1 \\
2 &= 2 \cdot 1 + 0
\end{aligned}
$$

Ignoring the last equation and rearranging the other equations so that the remainders are on the left side we get the following:

$$
\begin{aligned}
1312 &= 4001 - 2689 \\
65 &= 2689 - 2 \cdot 1312 \\
12 &= 1312 - 20 \cdot 65 \\
5 &= 65 - 5 \cdot 12 \\
2 &= 12 - 2 \cdot 5 \\
1 &= 5 - 2 \cdot 2
\end{aligned}
$$

Back substituting through the above equations we have the following:

$$
\begin{aligned}
1 &= 5 - 2 \cdot 2 \\
&= 5 - 2 \cdot (12 - 2 \cdot 5) \\
&= 5 \cdot 5 - 2 \cdot 12 \\
&= 5 \cdot (65 - 5 \cdot 12) - 2 \cdot (1312 - 20 \cdot 65) \\
&= 45 \cdot 65 - 25 \cdot 12 - 2 \cdot 1312 \\
&= 45 \cdot (2689 - 2 \cdot 1312) - 25 \cdot (1312 - 20 \cdot 65) - 2 \cdot (4001 - 2689) \\
&= -2 \cdot 4001 + 47 \cdot 2689 - 115 \cdot 1312 + 500 \cdot 65 \\
&= -2 \cdot 4001 + 47 \cdot 2689 - 115 \cdot (4001 - 2689) + 500 \cdot (2689 - 2 \cdot 1312) \\
&= -117 \cdot 4001 + 662 \cdot 2689 - 1000 \cdot 1312 \\
&= -117 \cdot 4001 + 662 \cdot 2689 - 1000 \cdot (4001 - 2689) \\
&= -1117 \cdot 4001 + 1662 \cdot 2689.
\end{aligned}
$$

This gives us the solution $x = -1117$ and $y = 1662$ to the equation $4001x + 2689y = 1$.

All the solutions are given by the formulas

$$x = -1117 - t(2689/1) = -1117 - 2689t$$

and

$$y = 1662 + t(4001/1) = 1662 + 4001t$$

where t is any integer.

Plugging in different values for t we get some more solutions:

$$\begin{array}{rcl} t=1 & \text{gives} & x=-3806 \text{ and } y=5663 \\ t=-1 & \text{gives} & x=1572 \text{ and } y=-2339 \\ t=2 & \text{gives} & x=-6495 \text{ and } y=9664 \\ t=-2 & \text{gives} & x=4261 \text{ and } y=-6340 \end{array}$$

(b) $864x + 468y = 36$

We have that $\gcd(864, 468) = 36$. Since 36 divides 36, we know that $864x + 468y = 36$ has integer solutions. Using the Euclidean algorithm we have that

$$\begin{array}{rcl} 864 & = & 1 \cdot 468 + 396 \\ 468 & = & 1 \cdot 396 + 72 \\ 396 & = & 5 \cdot 72 + 36 \\ 72 & = & 2 \cdot 36 \end{array}$$

Ignoring the last equation and rearranging the other equations so that the remainders are on the left side we get the following:

$$\begin{array}{rcl} 396 & = & 864 - 468 \\ 72 & = & 468 - 396 \\ 36 & = & 396 - 5 \cdot 72 \end{array}$$

Back substituting through the above equations we have the following:

$$
\begin{aligned}
36 &= 396 - 5 \cdot 72 \\
&= (864 - 468) - 5 \cdot (468 - 396) \\
&= 864 - 6 \cdot 468 + 5 \cdot 396 \\
&= 864 - 6 \cdot 468 + 5 \cdot (864 - 468) \\
&= 6 \cdot 864 - 11 \cdot 468.
\end{aligned}
$$

This gives us the solution $x = 6$ and $y = -11$ to the equation $864x + 468y = 36$.

All the solutions are given by the formulas

$$x = 6 - t(468/36) = 6 - 13t$$

and

$$y = -11 + t(864/36) = -11 + 24t$$

where t is any integer.

Plugging in different values for t we get some more solutions:

$$
\begin{array}{rll}
t = 1 & \text{gives} & x = -7 \text{ and } y = 13 \\
t = -1 & \text{gives} & x = 19 \text{ and } y = -35 \\
t = 2 & \text{gives} & x = -20 \text{ and } y = 37 \\
t = -2 & \text{gives} & x = 32 \text{ and } y = -59
\end{array}
$$

(c) $5x + 3y = 7$

Note that $\gcd(5, 3) = 1$. Since 1 divides 7 there exist integer solutions to $5x + 3y = 7$. To find these solutions we first find a solution to $5x + 3y = \gcd(5, 3) = 1$. One can use the Euclidean algorithm to do this. You should do this step if you need the practice. We have that $5 \cdot (-1) + 3 \cdot (2) = 1$. Now multiply the equation by 7 to get $5 \cdot (-7) + 3 \cdot (14) = 7$. Hence a solution to $5x + 3y = 7$ is given by $x = -7$ and $y = 14$.

All the solutions are given by the formulas

$$x = -7 - t(3/1) = -7 - 3t$$

and

$$y = 14 + t(5/1) = 14 + 5t$$

where t is any integer.

Plugging in different values for t we get some more solutions:

$$\begin{array}{rll} t = 1 & \text{gives} & x = -10 \text{ and } y = 19 \\ t = -1 & \text{gives} & x = -4 \text{ and } y = 9 \\ t = 2 & \text{gives} & x = -13 \text{ and } y = 24 \\ t = -2 & \text{gives} & x = -1 \text{ and } y = 4 \end{array}$$

(d) $1819x + 3587y = 17$

Note that $\gcd(1819, 3587) = 17$. Since 17 divides 17, we know that $1819x + 3587y = 17$ has integer solutions. Using the Euclidean algorithm we have that

$$\begin{array}{rcl} 3587 & = & 1 \cdot 1819 + 1768 \\ 1819 & = & 1 \cdot 1768 + 51 \\ 1768 & = & 34 \cdot 51 + 34 \\ 51 & = & 1 \cdot 34 + 17 \\ 34 & = & 2 \cdot 17 \end{array}$$

Ignoring the last equation and rearranging the other equations so that the remainders are on the left side we get the following:

$$\begin{array}{rcl} 1768 & = & 3587 - 1819 \\ 51 & = & 1819 - 1 \cdot 1768 \\ 34 & = & 1768 - 34 \cdot 51 \\ 17 & = & 51 - 1 \cdot 34 \end{array}$$

Back substituting through the above equations we have the following:

$$
\begin{aligned}
17 &= 51 - 1 \cdot 34 \\
&= (1819 - 1768) - (1768 - 34 \cdot 51) \\
&= 1819 - 2 \cdot 1768 + 34 \cdot 51 \\
&= 1819 - 2 \cdot (3587 - 1819) + 34 \cdot (1819 - 1768) \\
&= 37 \cdot 1819 - 2 \cdot 3587 - 34 \cdot 1768 \\
&= 37 \cdot 1819 - 2 \cdot 3587 - 34 \cdot (3587 - 1819) \\
&= 71 \cdot 1819 - 36 \cdot 3587
\end{aligned}
$$

This gives us the solution $x = 71$ and $y = -36$ to the equation $1819x + 3587y = 17$.

All the solutions are given by the formulas

$$x = 71 - t(3587/17) = 71 - 211t$$

and

$$y = -36 + t(1819/17) = -36 + 107t$$

where t is any integer.

Plugging in different values for t we get some more solutions:

$$
\begin{aligned}
t = 1 &\quad \text{gives} \quad x = -140 \text{ and } y = 71 \\
t = -1 &\quad \text{gives} \quad x = 282 \text{ and } y = -143 \\
t = 2 &\quad \text{gives} \quad x = -351 \text{ and } y = 178 \\
t = -2 &\quad \text{gives} \quad x = 493 \text{ and } y = -250
\end{aligned}
$$

(e) $10x + 105y = 101$

Note that $\gcd(10, 105) = 5$ and 5 does not divide 101. Hence $10x + 105y = 101$ does not have any integer solutions.

(f) $39x + 17y = 22$

Note that $\gcd(39, 17) = 1$ and 1 divides 22. Hence $39x + 17y = 22$ has integer solutions. We first find a solution to $39x + 17y =$

$\gcd(39, 17) = 1$ using the Euclidean algorithm. We then multiply that solution by 22 to get a solution to $39x + 17y = 22$.
We have that

$$\begin{aligned} 39 &= 2 \cdot 17 + 5 \\ 17 &= 3 \cdot 5 + 2 \\ 5 &= 2 \cdot 2 + 1 \\ 2 &= 2 \cdot 1 + 0 \end{aligned}$$

Rearranging the equations as in the above problems, we have that

$$\begin{aligned} 5 &= 39 - 2 \cdot 17 \\ 2 &= 17 - 3 \cdot 5 \\ 1 &= 5 - 2 \cdot 2 \end{aligned}$$

Hence

$$\begin{aligned} 1 &= 5 - 2 \cdot 2 \\ &= (39 - 2 \cdot 17) - 2 \cdot (17 - 3 \cdot 5) \\ &= 39 - 4 \cdot 17 + 6 \cdot 5 \\ &= 39 - 4 \cdot 17 + 6 \cdot (39 - 2 \cdot 17) \\ &= 7 \cdot 39 - 16 \cdot 17 \end{aligned}$$

Hence, a solution to $39 \cdot 7 + 17 \cdot (-16) = 1$. Multiplying the equation by 22 we get that $39 \cdot 154 + 17 \cdot (-352) = 22$. Hence $x = 154$ and $y = -352$ is an integer solution to the equation $39x + 17y = 22$.
All the solutions are given by the formulas

$$x = 154 - t(17/1) = 154 - 17t$$

and

$$y = -352 + t(39/1) = -352 + 39t$$

where t is any integer.

Plugging in different values for t we get some more solutions:

$$\begin{array}{rll} t = 1 & \text{gives} & x = 137 \text{ and } y = -313 \\ t = -1 & \text{gives} & x = 171 \text{ and } y = -391 \\ t = 2 & \text{gives} & x = 120 \text{ and } y = -274 \\ t = -2 & \text{gives} & x = 188 \text{ and } y = -430 \end{array}$$

(g) $3x + 18y = 9$

Note that $\gcd(3, 18) = 3$ and 3 divides 9. Hence $3x + 18y = 9$ has integer solutions. Note here that when one tries to do the Euclidean algorithm the process stops after one step:

$$18 = 6 \cdot 3 + 0$$

In this problem, 3 divides 18, so we know right away that $\gcd(3, 18) = 3$. How can we solve the linear equation though? Well, these kinds of problems are all of the same form. That is, the problem is of the form:

$$ax + aqy = ak$$

That is, the coefficient a divides both the b term and the c term. In this case, one can just make $x = k$ and $y = 0$. For our problem this is $x = 3$ and $y = 0$. That is, $3 \cdot 3 + 18 \cdot 0 = 9$.

Then the general solution to the equation is given by the formulas

$$x = 3 - t(18/3) = 3 - 6t$$

and

$$y = 0 + t(3/3) = t$$

where t is any integer.

Plugging in different values for t we get some more solutions:

$$\begin{array}{rll} t = 0 & \text{gives} & x = 3 \text{ and } y = 0 \\ t = 1 & \text{gives} & x = -3 \text{ and } y = 1 \\ t = -1 & \text{gives} & x = 9 \text{ and } y = -1 \\ t = 2 & \text{gives} & x = -9 \text{ and } y = 2 \\ t = -2 & \text{gives} & x = 15 \text{ and } y = -2 \end{array}$$

5. *Suppose that* a, b, x, y *are integers with* a *and* b *not both zero. Prove that* $\gcd(a, b)$ *divides* $ax + by$.

   Solution: Let $d = \gcd(a, b)$. Then $d|a$ and $d|b$. Hence $ds = a$ and $dt = b$ for some integers s and t. Therefore, $ax + by = dsx + dty = d(sx + ty)$. Hence $d|(ax + by)$.

6. *Prove that no integers* x *and* y *exist such that* $x - y = 200$ *and* $\gcd(x, y) = 3$.

   Solution: Suppose that there exist integers x and y with $x - y = 200$ and $\gcd(x, y) = 3$. Then $3s = x$ and $3t = y$ for some integers s and t because $3|x$ and $3|y$. So

$$200 = x - y = 3s - 3t = 3(s - t).$$

   Hence 3 would divide 200. But 3 does not divide 200. This is a contradiction. Hence no such integers exist.

7. Let a and b be integers, $a > 0$, $b > 0$, and $d = \gcd(a, b)$. Prove that $a|b$ if and only if $d = a$.

   Solution: Suppose that $a|b$. Since $a > 0$ and $a|a$ and $a|b$, we have that a is a positive common divisor of a and b. Since d is the largest common divisor of a and b we know that $a \leq d$. Furthermore, since d is a divisor of a and both a and d are positive, we have that $d \leq a$. Combining $a \leq d$ and $d \leq a$ we get that $d = a$.

   Conversely suppose that $d = a$. Note $d|b$ since d is the greatest common divisor of a and b. So $a|b$ since $a = d$.

8. Let a and b be integers such that $\gcd(a, 4) = 2$ and $\gcd(b, 4) = 2$. Prove that $\gcd(a + b, 4) = 4$.

   Solution: Note that 2 divides a since $\gcd(a, 4) = 2$. Thus $a = 2s$ for some integer s. Also, 2 divides b since $\gcd(b, 4) = 2$. Thus $b = 2t$ for some integer t.

   Note that 4 does not divide a since if it did then $\gcd(a, 4) = 4$ (since 4 would then be a common divisor of both a and 4). This isn't true because we assumed that $\gcd(a, 4) = 2$. Therefore, s must be odd. Thus $s = 2x + 1$ for some integer x.

Note that 4 does not divide b since if it did then $\gcd(b,4) = 4$ (since 4 would then be a common divisor of both b and 4). This isn't true because we assumed that $\gcd(b,4) = 2$. Therefore, t must be odd. Thus $t = 2y + 1$ for some integer y.

Therefore, $a + b = 2s + 2t = 2(2x + 1) + 2(2y + 1) = 4(x + y + 1)$. So 4 divides $a + b$. Therefore, $\gcd(a + b, 4) = 4$.

9. Suppose that x, y, z are integers with $x \neq 0$. Prove that $x|yz$ if and only if $\left.\frac{x}{\gcd(x,y)}\right| z$.

   Solution: Suppose that $x|yz$. Then $xk = yz$ for some integer k. Let $d = \gcd(x,y)$.

   Note that $d|x$ and $d|y$. Therefore, x/d and y/d are both integers. Dividing the equation $xk = yz$ by d gives that $(x/d)k = (y/d) \cdot z$. In class we showed that $\gcd(x/d, y/d) = 1$. Hence since x/d divides the product $(y/d) \cdot z$ and $\gcd(x/d, y/d) = 1$ we must have that $(x/d)|z$. This is what we wanted to prove.

   Conversely, suppose that $(x/d)|z$. Then $(x/d)k = z$ for some integer k. Since $d|y$ we must have that $ds = y$ for some integer s. Multiplying the equation $(x/d) \cdot k = z$ by y gives $(x/d) \cdot k \cdot y = yz$. Since $y/d = s$ we have that $x \cdot k \cdot s = yz$. Hence $x|yz$.

10. Let a, b, c be integers with $a \neq 0$ and $b \neq 0$. Prove that if $a|c$, $b|c$, and $\gcd(a,b) = 1$, then $ab|c$.

    Solution: Since $a|c$ we have that $ax = c$ for some integer x. Since $b|c$ we have that $by = c$ for some integer y. Therefore

    $$ax = c = by.$$

    Therefore, $b|ax$. Since $\gcd(b,a) = 1$ we must have that $b|x$. Therefore, $bk = x$ for some integer k. Hence

    $$c = ax = a(bk) = (ab)k.$$

    Therefore, $ab|c$.

11. Let a, b, c, x be integers with a and b not both zero and $x \neq 0$. Prove that if $\gcd(a,b) = 1$, $x|a$, and $x|bc$, then $x|c$.

Solution # 1: Since $x|a$ and $x|bc$ we have that $a = xk$ and $bc = xg$ where k and g are integers. Since $\gcd(a, b) = 1$, there exist integers x_0 and y_0 such that $ax_0+by_0 = 1$. Multiplying by c gives us $acx_0+bcy_0 = c$. Substituting the equations from the first sentence of this proof we have that $xkcx_0 + xgy_0 = c$. Thus, $x(kcx_0 + gy_0) = c$. This shows that $x|c$.

Solution # 2: Suppose that d is a positive common divisor of x and b. [We will show that $d = 1$. This will imply that $\gcd(x, b) = 1$.] By definition $d|x$ and $d|b$. Since $d|x$ and $x|a$ we have that $d|a$. Hence d is a positive common divisor of a and b. Therefore $d = 1$ because $\gcd(a, b) = 1$ by assumption. Thus, the only positive divisor of x and b is the integer 1. Hence $\gcd(x, b) = 1$.

Since $x|bc$ and $\gcd(x, b) = 1$ we have that $x|c$.

12. Suppose that a and b are integers, not both zero. Suppose that there exist integers x and y with $ax + by = 1$. Prove that $\gcd(a, b) = 1$.

    Solution: Let $d = \gcd(a, b)$. Thus d is a positive common divisor of a and b. Since $d|a$ and $d|b$ we have that $d|(ax + by)$. Hence $d|1$. Thus $d = 1$ because d is positive.

13. Show that the following is not necessarily true: If a, b, c, x, y are integers and $ax + by = c$ then $\gcd(a, b) = c$.

    Solution: Try $a = 1$, $b = 2$, $c = 2$, $x = 4$, and $y = -1$. Note that $\gcd(a, b) = 1$.