1. A set $S$ and a relation $\sim$ on $S$ is given. For each example, check if $\sim$ is (i) reflexive, (ii) symmetric, and/or (iii) transitive. If $\sim$ satisfies the property that you are checking, then prove it. If $\sim$ does not satisfy the property that you are checking, then give an example to show it.

(a) $S = \mathbb{R}$ where $a \sim b$ if and only if $a \leq b$.

**Solution:**
(i) Yes, $\sim$ is reflexive. Proof: Let $a \in \mathbb{R}$. Then $a \leq a$. So $a \sim a$.
(ii) No, $\sim$ is not symmetric. Counterexample: $3 \leq 4$, but $4 \not\leq 3$. That is, $3 \sim 4$ but $4 \not\sim 3$.
(iii) Yes, $\sim$ is transitive. Proof: Let $a, b, c \in \mathbb{R}$ and suppose that $a \sim b$ and $b \sim c$. Then $a \leq b$ and $b \leq c$. So $a \leq c$. Thus $a \sim c$.

(b) $S = \mathbb{R}$ where $a \sim b$ if and only if $|a| = |b|$.

**Solution:**
(i) Yes, $\sim$ is reflexive. Proof: Let $a \in \mathbb{R}$. Then $|a| = |a|$. So $a \sim a$.
(ii) Yes, $\sim$ is symmetric. Proof: Let $a, b \in \mathbb{R}$ and suppose that $a \sim b$. Then $|a| = |b|$. So $|b| = |a|$. Thus $b \sim a$.
(iii) Yes, $\sim$ is transitive. Proof: Let $a, b, c \in \mathbb{R}$ and suppose that $a \sim b$ and $b \sim c$. Then $|a| = |b|$ and $|b| = |c|$. So $|a| = |c|$. Thus $a \sim c$.

(c) $S = \mathbb{Z}$ where $a \sim b$ if and only if $a|b$.

**Solution:**
(i) Yes, $\sim$ is reflexive. Proof: Let $a \in \mathbb{Z}$. Then $a(1) = a$. Hence $a|a$. So $a \sim a$.
(ii) No, $\sim$ is not symmetric. Counterexample: $3|6$, but $6 \not| 3$.
(iii) Yes, $\sim$ is transitive. Proof: Let $a, b, c \in \mathbb{Z}$. Suppose that $a \sim b$ and $b \sim c$. Then $a|b$ and $b|c$. Thus there exists $k, m \in \mathbb{Z}$ such that $ak = b$ and $bm = c$. Then $c = bm = (ak)m = a(km)$. So $a|c$. Thus $a \sim c$.

(d) $S$ is the set of subsets of $\mathbb{N}$ where $A \sim B$ if and only if $A \subseteq B$.

Some examples of elements of $S$ are $\{1, 10, 199\}$, $\{2, 7, 10\}$, and $\{2, 10, 3, 7\}$. Note that $\{2, 7, 10\} \sim \{2, 10, 3, 7\}$.
Solution:
(i) Yes, ∼ is reflexive. Proof: \( A \subseteq A \) for all \( A \in S \).
(ii) No, ∼ is not symmetric. Counterexample: \( \{3\} \subseteq \{3, 42\} \), but \( \{3, 42\} \not\subseteq \{3\} \).
(iii) Yes, ∼ is transitive. Proof: Let \( A, B, C \in S \) with \( A \sim B \) and \( B \sim C \). Then \( A \subseteq B \) and \( B \subseteq C \). We want to show that \( A \subseteq C \). Let \( x \in A \). Since \( A \subseteq B \), we have that \( x \in B \). Since \( B \subseteq C \) we have that \( x \in C \). So \( A \subseteq C \) and thus \( A \sim C \).

2. Consider the set \( S = \mathbb{R} \) where \( x \sim y \) if and only if \( x^2 = y^2 \).

(a) Find all the numbers that are related to \( x = 1 \). Repeat this exercise for \( x = \sqrt{2} \) and \( x = 0 \).

Solution:
1 \( \sim \) 1 since \( 1^2 = 1^2 \). We also have 1 \( \sim \) \((-1)\) since \( 1^2 = (-1)^2 \). There are no other elements related to 1.
\( \sqrt{2} \sim \sqrt{2} \) since \((\sqrt{2})^2 = (\sqrt{2})^2 \). We also have \( \sqrt{2} \sim (-\sqrt{2}) \) since \((\sqrt{2})^2 = (-\sqrt{2})^2 \). There are no other elements related to \( \sqrt{2} \).
0 \( \sim \) 0 since \( 0^2 = 0^2 \). There are no other elements related to 0.

(b) Prove that ∼ is an equivalence relation on \( S \).

Solution:
Proof. Reflexive: We know that \( x^2 = x^2 \) for all real numbers \( x \). Therefore \( x \sim x \) for all real numbers \( x \). So ∼ is reflexive.
Symmetric: Let \( x, y \in \mathbb{R} \). Suppose that \( x \sim y \).
Since \( x \sim y \) we have that \( x^2 = y^2 \).
So \( y^2 = x^2 \).
Therefore \( y \sim x \).
Transitive Let \( x, y, z \in \mathbb{R} \). Suppose that \( x \sim y \) and \( y \sim z \).
Since \( x \sim y \) we have that \( x^2 = y^2 \).
Since \( y \sim z \) we have that \( y^2 = z^2 \).
So \( x^2 = y^2 = z^2 \).
Therefore \( x \sim z \).

(c) Draw a number line. Draw a picture of the equivalence class of 1. Repeat this for \( x = 0 \), \( x = \sqrt{6} \), \( x = -3 \).

Solution: Please draw a picture.
(d) Describe the elements of $S/\sim$.

**Solution:**

If $x \neq 0$, then the equivalence class of $x$ is $[x] = \{-x, x\}$. The equivalence class of $0$ is $[0] = \{0\}$.

3. Consider the set $S = \mathbb{Z}$ where $x \sim y$ if and only if $2|(x+y)$.

(a) List six numbers that are related to $x = 2$.

**Solution:**

$2 \sim (-4)$ since $2|(2+(-4))$.
$2 \sim (-2)$ since $2|(2+(-2))$.
$2 \sim (0)$ since $2|(2+(0))$.
$2 \sim (2)$ since $2|(2+(2))$.
$2 \sim (4)$ since $2|(2+(4))$.
$2 \sim (6)$ since $2|(2+(6))$.

(b) Prove that $\sim$ is an equivalence relation on $S$.

**Proof.** Reflexive: Let $x \in \mathbb{Z}$.

Since $2|2x$ we have that $2|(x+x)$.

So $x \sim x$.

Symmetric: Let $x, y \in \mathbb{Z}$ and suppose that $x \sim y$.

Thus $2|(x+y)$.

So $2|(y+x)$.

So $y \sim x$.

Transitive: Let $x, y, z \in \mathbb{Z}$ and suppose that $x \sim y$ and $y \sim z$.

Therefore $2|(x+y)$ and $2|(y+z)$.

So there exist $k, \ell \in \mathbb{Z}$ such that $2k = x+y$ and $2\ell = y+z$.

Add these equations to get $2k + 2\ell = x + 2y + z$.

Subtract $2y$ from both sides to get $2(k + \ell - y) = x + z$.

Note that $k + \ell - y \in \mathbb{Z}$, because $k, \ell, y \in \mathbb{Z}$ and $\mathbb{Z}$ is closed under addition and subtraction.

So $2|(x+z)$.

So $x \sim z$. 

\qed
(c) Draw a picture of the set of integers. Next, circle the numbers
that are in the equivalence class of \(-3\).

**Solution:** Draw a picture and circle these numbers:
\[ \ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots \]

(d) Describe the elements of \( S/\sim \). Draw a picture of several equivalence classes.

**Solution:** Draw a picture of the following:
\[ \bar{0} = \{ \ldots, -6, -4, -2, 0, 2, 4, 6, \ldots \} = -2 = \bar{2} = 4 = \bar{-4} = \ldots \\
\bar{1} = \{ \ldots, -7, -5, -3, -1, 1, 3, 5, 7, \ldots \} = -1 = \bar{3} = -3 = \bar{-5} = \ldots \]

So \( S/\sim \) is equal to \( \{\bar{0}, \bar{1}\} \). That is, one equivalence class is the set of all odd numbers; the other equivalence class is the set of all even numbers.

4. Show that the operation \( \bar{a} \oplus \bar{b} = \bar{a}^2 + \bar{b}^2 \) is a well-defined operation for \( \mathbb{Z}_n \). Here \( \bar{a}^2 \) means \( \bar{a} \cdot \bar{a} \). For example, in \( \mathbb{Z}_4 \) we have that
\[ \bar{2} \oplus \bar{3} = \bar{2} \cdot \bar{2} + \bar{3} \cdot \bar{3} = \bar{4} + \bar{9} = \bar{1}. \]

**Proof.**

1) Let \( \bar{a}, \bar{b} \in \mathbb{Z}_n \) where \( a, b \in \mathbb{Z} \).

Then
\[ \bar{a} \oplus \bar{b} = \bar{a}^2 + \bar{b}^2 = \bar{a^2 + b^2} = \bar{a}^2 + \bar{b^2}. \]

Since \( a, b \in \mathbb{Z} \) we have that \( a^2 + b^2 \in \mathbb{Z} \).

Therefore, \( \bar{a} \oplus \bar{b} = \bar{a^2 + b^2} \in \mathbb{Z}_n \).

So \( \mathbb{Z}_n \) is closed under the operation \( \oplus \).

2) Suppose that \( a_1, a_2, b_1, b_2 \in \mathbb{Z} \) such that \( \bar{a_1} = \bar{a_2} \) and \( \bar{b_1} = \bar{b_2} \). We need to show that \( \bar{a_1} \oplus \bar{b_1} = \bar{a_2} \oplus \bar{b_2} \).

From class we had a theorem that says that if \( \bar{x} = \bar{y} \) and \( \bar{w} = \bar{z} \), then
\( \bar{x} + \bar{w} = \bar{y} + \bar{z} \) and \( \bar{x} \cdot \bar{w} = \bar{y} \cdot \bar{z} \).

Repeatedly using the above theorem we get the following.

We have that \( \bar{a_1} \cdot \bar{a_1} = \bar{a_2} \cdot \bar{a_2} \) by multiplying the equations \( \bar{a_1} = \bar{a_2} \) and \( \bar{a_1} = \bar{a_2} \).

Similarly, \( \bar{b_1} \cdot \bar{b_1} = \bar{b_2} \cdot \bar{b_2} \) by multiplying the equations \( \bar{b_1} = \bar{b_2} \) and \( \bar{b_1} = \bar{b_2} \).
Adding the two equations above we get that \( \overline{a_1 \cdot a_1} + \overline{b_1 \cdot b_1} = \overline{a_2 \cdot a_2} + \overline{b_2 \cdot b_2} \).

Therefore, \( \overline{a_1} \oplus \overline{b_1} = \overline{a_2} \oplus \overline{b_2} \).

Thus \( \oplus \) is a well-defined operation on \( \mathbb{Z}_n \).

5. Given two integers \( a \) and \( b \), let \( \min(a, b) \) denote the minimum (smaller) of \( a \) and \( b \). Let \( n \) be an integer with \( n \geq 2 \). Is the operation \( a \oplus b = \min(a, b) \) a well-defined operation on \( \mathbb{Z}_n \)?

**Solution:** This operation is not well-defined. For example, consider \( n = 4 \). In \( \mathbb{Z}_4 \) we have that \( \overline{0} = \overline{8} \) and \( \overline{1} = \overline{5} \). Thus, for the operation to be well-defined we would need \( \overline{0} \oplus \overline{1} = \overline{8} \oplus \overline{5} \). However, \( \overline{0} \oplus \overline{1} = \min(0, 1) = \overline{0} \) and \( \overline{8} \oplus \overline{5} = \min(8, 5) = \overline{5} \). But \( \overline{0} \neq \overline{5} \) in \( \mathbb{Z}_4 \).

6. (a) Show that the operation \( \frac{a}{b} \oplus \frac{c}{d} = \frac{ad}{bc} \) is not a well-defined operation on \( \mathbb{Q} \). (b) Is the operation well-defined on \( \mathbb{Q} - \{0\} \)?

(a) Show that the operation \( \frac{a}{b} \oplus \frac{c}{d} = \frac{ad}{bc} \) is not a well-defined operation on \( \mathbb{Q} \).

**Solution:** We have that \( \frac{5}{2}, \frac{0}{1} \in \mathbb{Q} \) however \( \frac{5}{2} \oplus \frac{0}{1} = \frac{5 \cdot 1}{2 \cdot 0} = \frac{5}{0} \notin \mathbb{Q} \).

Hence \( \mathbb{Q} \) is not closed under \( \oplus \) and the operation is not well-defined.

(b) Is the operation well-defined on \( \mathbb{Q} \setminus \{0\} \)?

**Solution:** Yes! Here is a proof.

**Proof.** 1) Let \( a, b, c, d \in \mathbb{Z} \) with \( a \neq 0, b \neq 0, c \neq 0, d \neq 0 \) so that \( \frac{a}{b}, \frac{c}{d} \in \mathbb{Q} - \{0\} \).

Since \( a \neq 0, b \neq 0, c \neq 0, d \neq 0 \) we have that \( ad \neq 0 \) and \( bc \neq 0 \).

Thus \( \frac{a}{b} \oplus \frac{c}{d} = \frac{ad}{bc} \in \mathbb{Q} - \{0\} \).

Therefore \( \mathbb{Q} - \{0\} \) is closed under the operation \( \oplus \).

2) Suppose further that we have \( e, f, g, h \in \mathbb{Z} \) with \( e \neq 0, f \neq 0, g \neq 0, h \neq 0 \) so that \( \frac{e}{f}, \frac{g}{h} \in \mathbb{Q} - \{0\} \).

Also assume that \( \frac{e}{f} = \frac{g}{h} \) and \( \frac{e}{f} = \frac{g}{h} \).

We want to show that \( \frac{e}{f} \oplus \frac{g}{h} = \frac{e}{f} \oplus \frac{g}{h} \).

We have that \( \frac{a}{b} \oplus \frac{c}{d} = \frac{ad}{bc} \) and \( \frac{e}{f} \oplus \frac{g}{h} = \frac{eh}{fg} \).
Since \( \frac{a}{b} = \frac{c}{d} \) we have that \( af = be \).

Since \( \frac{c}{a} = \frac{d}{b} \) we have that \( ch = dg \).

Multiplying \( af = be \) by \( dg = ch \) we get \( afdg = bech \).

Rearranging we get \( (ad)(fg) = (bc)(eh) \).

Therefore, \( \frac{ad}{bc} = \frac{eh}{fg} \).

So \( \frac{a}{b} \oplus \frac{c}{d} = \frac{e}{f} \oplus \frac{g}{h} \).

Thus, the operation is well-defined.

\( \square \)

7. Is the operation \( \overline{a} \oplus \overline{b} = \overline{a+b} \) a well-defined operation on \( \mathbb{Z}_n \)?

**Solution:** There are two issues with this operation.

One issue is as follows. As an example, consider \( n = 4 \). In \( \mathbb{Z}_4 \) we have that \( \overline{1} = \overline{5} \). Thus, for the operation to be well-defined we must have that \( \overline{2} \oplus \overline{1} = \overline{2} \oplus \overline{5} \). However, \( \overline{2} \oplus \overline{1} = \overline{0} = \overline{2} \) and \( \overline{2} \oplus \overline{5} = \overline{32} = \overline{0} \).

And \( \overline{2} \neq \overline{0} \) in \( \mathbb{Z}_4 \).

Another issue is when \( b \) is a negative integer. For example, in \( \mathbb{Z}_4 \) suppose we want to calculate \( \overline{2} \oplus \overline{-1} \). What does this mean? The formula says that it is \( \overline{2-1} \). But what is that in \( \mathbb{Z}_4 \)? In fact there is no way to make sense of \( \overline{2} \cdot \overline{-1} \) in \( \mathbb{Z}_4 \) because there is no multiplicative inverse for \( \overline{2} \) in \( \mathbb{Z}_4 \). (Why?) Because there is no \( \overline{x} \in \mathbb{Z}_4 \) with \( \overline{x} \cdot \overline{2} = \overline{1} \).

We can check:

\begin{align*}
&\overline{0} \cdot \overline{2} = \overline{0} \neq \overline{1} \\
&\overline{1} \cdot \overline{2} = \overline{2} \neq \overline{1} \\
&\overline{2} \cdot \overline{2} = \overline{4} = \overline{0} \neq \overline{1} \\
&\overline{3} \cdot \overline{2} = \overline{6} = \overline{2} \neq \overline{1} \\
\end{align*}

Thus there is no way to define \( \overline{2} \oplus \overline{-1} \) in \( \mathbb{Z}_4 \).

8. (Constructing the integers from the natural numbers) Let \( S = \mathbb{N} \times \mathbb{N} \).

Define the relation \( \sim \) on \( S \) where \( (a,b) \sim (c,d) \) if and only if \( a+d = b+c \).

(a) Is \( (3,6) \sim (7,10) \)?

**Solution:** Yes, because \( 3 + 10 = 6 + 7 \).

(b) Is \( (1,1) \sim (3,5) \)?

**Solution:** No, because \( 1 + 5 \neq 1 + 3 \).
(c) Prove that $\sim$ is an equivalence relation.

\textit{Proof.} \textbf{Reflexive:} Let $(a, b) \in \mathbb{N} \times \mathbb{N}$. Then $a + b = b + a$. So $(a, b) \sim (a, b)$.

\textbf{Symmetric:} Let $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$. Suppose $(a, b) \sim (c, d)$. We know that $a + d = b + c$, because $(a, b) \sim (c, d)$. So $c + b = d + a$. So $(c, d) \sim (a, b)$.

\textbf{Transitive:} Let $(a, b), (c, d), (e, f) \in \mathbb{N} \times \mathbb{N}$. Suppose that $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. We know that $a + d = b + c$ and $c + f = d + e$, because $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$. Add these two equations to get $a + c + d + f = b + c + d + e$. Subtract $c + d$ from both sides to get $a + f = b + e$. So $(a, b) \sim (e, f)$.

Therefore, $\sim$ is an equivalence relation, because it is reflexive, symmetric, and transitive.

\hfill \square

(d) List five elements from each of the following equivalence classes: $(1, 1), (1, 2), (5, 12)$.

\textbf{Solution:} Some possible answers:

$(2, 2), (3, 3), (4, 4), (5, 5), (47, 47) \in (1, 1)$.
$(2, 3), (3, 4), (4, 5), (5, 6), (47, 48) \in (1, 2)$.
$(2, 9), (3, 10), (4, 11), (5, 12), (47, 56) \in (5, 12)$.

(e) Define the operation $(a, b) \oplus (c, d) = (a + c, b + d)$. Prove that $\oplus$ is well-defined on the set of equivalence classes.

\textit{Proof.} 1) Consider two equivalence classes $(a, b)$ and $(c, d)$ where $(a, b), (c, d) \in \mathbb{N} \times \mathbb{N}$. Then $a + c$ and $b + d$ are both in $\mathbb{N}$ because $\mathbb{N}$ is closed under addition.
Thus, \((a, b) \oplus (c, d) = (a + c, b + d)\) is a valid equivalence class in \(\mathbb{N} \times \mathbb{N} / \sim\).

2) Now suppose that \((a, b), (c, d), (e, f),\) and \((g, h)\) are equivalence classes of \(\mathbb{N} \times \mathbb{N} / \sim\).

Further suppose that \((a, b) = (e, f)\) and \((c, d) = (g, h)\).

We need to show that \((a, b) \oplus (c, d) = (e, f) \oplus (g, h)\).

We have that \(a + f = b + e\) since \((a, b) = (e, f)\).

We also have that \(c + h = d + g\) since \((c, d) = (g, h)\).

Adding these two equations gives \(a + f + c + h = b + e + d + g\).

Rearranging gives \((a + c) + (f + h) = (b + d) + (e + g)\).

Therefore, \((a + c, b + d) = (e + g, f + h)\).

Hence \((a, b) \oplus (c, d) = (e, f) \oplus (g, h)\).

The above arguments show that \(\oplus\) is a well-defined operation on the equivalence classes of \(\mathbb{N} \times \mathbb{N} / \sim\).

9. (Constructing the rational numbers from the integers) Let \(S = \mathbb{Z} \times (\mathbb{Z} - \{0\})\). Define the relation \(\sim\) on \(S\) where \((a, b) \sim (c, d)\) if and only if \(ad = bc\).

(a) Is \((1, 5) \sim (-3, -15)\) ?

**Solution:** Yes, because \(1(-15) = 5(-3)\).

(b) Is \((-1, 1) \sim (2, 3)\) ?

**Solution:** No, because \((-1)(3) \neq 1(2)\).

(c) Prove that \(\sim\) is an equivalence relation.

**Proof.**

**Reflexive:** Let \((a, b) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})\).

Then \(ab = ba\).

So \((a, b) \sim (a, b)\).

**Symmetric:** Let \((a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})\).

Suppose \((a, b) \sim (c, d)\).

We know that \(ad = bc\), because \((a, b) \sim (c, d)\).

So \(cb = da\).
Hence $(c, d) \sim (a, b)$.

**Transitive:** Let $(a, b), (c, d), (e, f) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$.
Suppose $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.
We know that $ad = bc$ and $cf = de$, because $(a, b) \sim (c, d)$ and $(c, d) \sim (e, f)$.
Multiply these two equations to get $adcf = bcde$.
Divide both sides by $c$ and then by $d$ to get $af = be$. (Note that $c, d \neq 0$ because $c, d \in \mathbb{Z} - \{0\}$, so it’s okay to divide by $c$ and by $d$.)
So $(a, b) \sim (e, f)$ since $af = be$.
Therefore, $\sim$ is an equivalence relation, because it is reflexive, symmetric, and transitive.

(d) List five elements from each of the following equivalence classes: $(1,1), (0,2), (2,3)$.

**Solution:** Some possible answers:
$(2,2), (3,3), (4,4), (5,5), (47,47) \in (1,1)$.
$(0,1), (0,2), (0,-1), (0,-2), (0,-47) \in (0,2)$.
$(2,3), (4,6), (6,9), (-2,-3), (-4,-6) \in (2,3)$.

(e) Define the operation $(a, b) \oplus (c, d) = (ad + bc, bd)$. Prove that $\oplus$ is well-defined on the set of equivalence classes.

**Proof.** 1) Consider two equivalence classes $(a, b)$ and $(c, d)$ where $(a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$.
Then $ad + bc \in \mathbb{Z}$ because $a, b, c, d \in \mathbb{Z}$ and the integers are closed under addition and multiplication.
Also, since $b, d \in \mathbb{Z} - \{0\}$ we have that $bd \neq 0$ and so $bd \in \mathbb{Z} - \{0\}$.
Thus $(ad + bc, bd) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})$ and $(a, b) \oplus (c, d) = (ad + bc, bd)$ is a valid equivalence class.

2) Now suppose that $(a, b), (c, d), (x, y), (w, z)$ are equivalence classes in $\mathbb{Z} \times (\mathbb{Z} - \{0\})/ \sim$.
Further suppose that $(a, b) = (x, y)$ and $(c, d) = (w, z)$. 
We need to show that \((a, b) \oplus (c, d) = (x, y) \oplus (w, z)\).
That is, we need to show that \([(ad + bc, bd)] = [(xz + yw, yz)]\).
The above is equivalent to showing that \((ad + bc)yz = bd(xz + yw)\).
Let’s do this.

Since \((a, b) = (x, y)\) we have that \(ay = bx\).
Since \((c, d) = (w, z)\) we have that \(cz = dw\).
Therefore, using the equations \(ay = bx\) and \(cz = dw\) we get that
\[
(ad + bc)yz = adyz + bcyz = (ay)(dz) + (cz)(by) = (bx)(dz) + (dw)(by) = bd(xz + yw).
\]

Thus, \([(ad + bc, bd)] = [(xz + yw, yz)]\).
Thus, the operation \(\oplus\) is well-defined on the equivalence classes of \(\mathbb{Z} \times (\mathbb{Z} - \{0\})/\sim\).

(f) Define the operation \((a, b) \odot (c, d) = (ac, bd)\). Prove that \(\odot\) is well-defined on the set of equivalence classes.

**Proof.** 1) Consider two equivalence classes \((a, b)\) and \((c, d)\) where \((a, b), (c, d) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})\).
Then \(ac \in \mathbb{Z}\) because \(a, c \in \mathbb{Z}\) and the integers are closed under multiplication.
Also, since \(b, d \in \mathbb{Z} - \{0\}\) we have that \(bd \neq 0\) and so \(bd \in \mathbb{Z} - \{0\}\).
Thus \((ac, bd) \in \mathbb{Z} \times (\mathbb{Z} - \{0\})\) and \((a, b) \odot (c, d) = (ac, bd)\) is a valid equivalence class.

2) Now suppose that \((a, b),(c, d),(x, y)\),and \((w, z)\) are equivalence classes in \(\mathbb{Z} \times (\mathbb{Z} - \{0\})/\sim\).
Further suppose that \((a, b) = (x, y)\) and \((c, d) = (w, z)\).
We need to show that \((a, b) \odot (c, d) = (x, y) \odot (w, z)\).
That is, we need to show that \([(ac, bd)] = [(xw, yz)]\).
The above is equivalent to showing that \((ac)(yz) = (bd)(xw)\).
Let’s do this.

Since \((a, b) = (x, y)\) we have that \(ay = bx\).

Since \((c, d) = (w, z)\) we have that \(cz = dw\).

Therefore, using the equations \(ay = bx\) and \(cz = dw\) we get that

\[(ac)(yz) = (ay)(cz) = (bx)(dw) = (bd)(xw).\]

Thus, \([\{(ac, bd)\}] = [(xw, yz)]\).

Therefore, the operation \(\odot\) is well-defined on the equivalence classes of \(\mathbb{Z} \times (\mathbb{Z} - \{0\}) / \sim\).