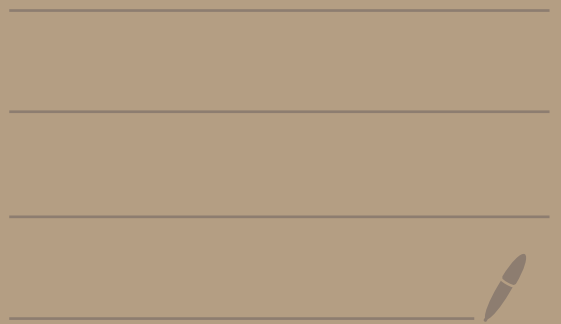


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HW 4 - Part 2

Solutions



① Let

$$A = \begin{pmatrix} 2 & -4 & 5 \\ -1 & 0 & 1 \\ 1 & -4 & 6 \end{pmatrix}$$

There are two possibilities,
Either A is invertible or A is
not invertible.

If A is invertible, then given
the system

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$$

we would get

$$A^{-1} A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix} .$$

But then the system

$$2x_1 - 4x_2 + 5x_3 = 4$$

$$-x_1 + x_3 = 2$$

$$x_1 - 4x_2 + 6x_3 = 7$$

would have a solution.

We are given though that it doesn't have a solution.

Thus, the possibility that A^{-1} exists cannot be true.

So, A^{-1} does not exist.

②

We are given that $B^2 = I$
and $A = PBQ$ and that
 P and Q are inverses.

P and Q are inverses means
that $PQ = QP = I$. [That is, $P^{-1} = Q$].

Let's show that $A^2 = I$,

We have that

$$A^2 = AA$$

$$= (PBQ)(PBQ)$$

$$= PB(QP)BQ$$

$QP = I$ \downarrow

$$= PBIBQ$$

$$= PBBQ$$

$$= PB^2Q = \downarrow$$

$$= PIQ$$

$$B^2 = I$$

$$= PQ$$

$$= I$$

$$PQ = I$$

$$So, A^2 = I.$$

③ Suppose that A is 3×3 and that

$$A^3 = O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We want to show that

$$(I - A)^{-1} = I + A + A^2.$$

Note that $X^{-1} = Y$ means that

$$XY = YX = I.$$

We have that

$$\begin{aligned} (I - A)(I + A + A^2) &= II + IA + IA^2 \\ &\quad - AI - AA - AA^2 \\ &= I + A + A^2 \\ &\quad - A - A^2 - A^3 \\ &= I - A^3 \\ &= I - O = I \end{aligned}$$

and

$$\begin{aligned} (I+A+A^2)(I-A) &= II - IA + AI \\ &\quad - AA + A^2I - A^2A \\ &= I - A + A - A^2 \\ &\quad + A^2 - A^3 \\ &= I - 0 \\ &= I. \end{aligned}$$

Therefore, since $(I-A)(I+A+A^2) = I$
and $(I+A+A^2)(I-A) = I$

we have that $I-A$ and
 $I+A+A^2$ are inverses and

thus,

$$(I-A)^{-1} = I+A+A^2.$$

④ Let A, C, D be $n \times n$ matrices and I be the $n \times n$ identity matrix. Suppose that $CA = I$ and $AD = I$.

We want to use the above information to prove that $C = D$.

Take $CA = I$ and multiply both sides by D on the right side.

Then

$$(CA)D = ID,$$

because
 $ID = D$

So,

$$C(AD) = D.$$

because
 $AD = I$

Thus,

$$CI = D$$

So,

$$C = D.$$

because
 $CI = C$

⑤ Suppose that A is $n \times n$
and $\vec{y}, \vec{x} \in \mathbb{R}^n$ with $\vec{x} \neq \vec{y}$.

Suppose further that $A\vec{x} = A\vec{y}$.

We must show that A is
not invertible.

We do this by ruling out the case
that A is invertible.

For if A is invertible then we
could multiply both sides of

$$A\vec{x} = A\vec{y} \text{ by } A^{-1} \text{ to get}$$

$$A^{-1}A\vec{x} = A^{-1}A\vec{y} \text{ which gives}$$

$$I\vec{x} = I\vec{y} \text{ which gives } \vec{x} = \vec{y}.$$

$$\text{But } \vec{x} \neq \vec{y}.$$

Therefore, we cannot have that A
is invertible under the given conditions.
So, A^{-1} does not exist.

⑥ Suppose A^{-1} exists.

Apply A^{-1} to $A\vec{x} = \vec{b}$ to get

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

which gives

$$I\vec{x} = A^{-1}\vec{b}$$

which yields

$$\vec{x} = A^{-1}\vec{b}.$$

Thus, if A^{-1} exists then $A\vec{x} = \vec{b}$ has exactly one solution $\vec{x} = A^{-1}\vec{b}$.

Therefore if $A\vec{x} = \vec{b}$ has infinitely many solutions then A^{-1} cannot exist.