2550
HW 4 — Part 2
Solutions

① Let

$$A = \begin{pmatrix} 2 & -4 & 5 \\ -1 & 0 & 1 \\ 1 & -4 & 6 \end{pmatrix}$$

There are two possibilities,
Either A is invertible or A is
not invertible.

If A is invertible, then given
the system

$$A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$$

we would get

$$A^{-1} A \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}$$

or

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix} = A^{-1} \begin{pmatrix} 4 \\ 2 \\ 7 \end{pmatrix}.$$

But then the system

$$\begin{aligned} 2x_1 - 4x_2 + 5x_3 &= 4 \\ -x_1 \qquad\quad + x_3 &= 2 \\ x_1 - 4x_2 + 6x_3 &= 7 \end{aligned}$$

would have a solution.

We are given though that it doesn't have a solution.

Thus, the possibility that A^{-1} exists cannot be true.

So, A^{-1} does not exist.

②

We are given that $B^2 = I$ and $A = PBQ$ and that P and Q are inverses.

P and Q are inverses means that $PQ = QP = I$. [That is, $P^{-1} = Q$].

Let's show that $A^2 = I$.

We have that

$$
\begin{aligned}
A^2 &= AA \\
&= (PBQ)(PBQ) \\
&= PB(QP)BQ \\
&\overset{QP=I}{=} PBIBQ \\
&= PBBQ \\
&= PB^2Q =
\end{aligned}
$$

$$= PIQ$$

$\boxed{B^2 = I}$

$$= PQ$$

$$= I$$

$\boxed{PQ = I}$

So, $A^2 = I$.

(3) Suppose that A is 3×3 and that

$$A^3 = O = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

We want to show that

$$(I-A)^{-1} = I + A + A^2.$$

Note that $X^{-1} = Y$ means that

$XY = YX = I$.

We have that

$$\begin{aligned}(I-A)(I+A+A^2) &= II + IA + IA^2 \\ &\quad - AI - AA - AA^2 \\ &= I + A + A^2 \\ &\quad - A - A^2 - A^3 \\ &= I - A^3 \\ &= I - O = I\end{aligned}$$

and

$$
\begin{aligned}
(I+A+A^2)(I-A) &= II - IA + AI \\
&\quad - AA + A^2I - A^2A \\
&= I - A + A - A^2 \\
&\quad + A^2 - A^3 \\
&= I - 0 \\
&= I.
\end{aligned}
$$

Therefore, since $(I-A)(I+A+A^2) = I$

and $(I+A+A^2)(I-A) = I$

we have that $I-A$ and $I+A+A^2$ are inverses and

thus,

$$(I-A)^{-1} = I + A + A^2.$$

4. Let A, C, D be $n \times n$ matrices and I be the $n \times n$ identity matrix. Suppose that $CA = I$ and $AD = I$. We want to use the above information to prove that $C = D$.

Take $CA = I$ and multiply both sides by D on the right side.

Then

$$(CA)D = ID.$$

because $ID = D$

So,

$$C(AD) = D.$$

because $AD = I$

Thus,

$$CI = D$$

because $CI = C$

So, $C = D$.

(5) Suppose that A is $n \times n$ and $\vec{y}, \vec{x} \in \mathbb{R}^n$ with $\vec{x} \neq \vec{y}$.

Suppose further that $A\vec{x} = A\vec{y}$.

We must show that A is not invertible.

We do this by ruling out the case that A is invertible.

For if A is invertible then we could multiply both sides ~~of~~ $A\vec{x} = A\vec{y}$ by A^{-1} to get

$A^{-1}A\vec{x} = A^{-1}A\vec{y}$ which gives

$I\vec{x} = I\vec{y}$ which gives $\vec{x} = \vec{y}$.

But $\vec{x} \neq \vec{y}$.

Therefore, we cannot have that A is invertible under the given conditions.

So, A^{-1} does not exist.

⑥ Suppose A^{-1} exists.

Apply A^{-1} to $A\vec{x} = \vec{b}$ to get

$$A^{-1}(A\vec{x}) = A^{-1}\vec{b}$$

which gives

$$I\vec{x} = A^{-1}\vec{b}$$

which yields

$$\vec{x} = A^{-1}\vec{b}.$$

Thus, if A^{-1} exists then $A\vec{x} = \vec{b}$ has exactly one solution $\vec{x} = A^{-1}\vec{b}$.

Therefore if $A\vec{x} = \vec{b}$ has infinitely many solutions then A^{-1} cannot exist.