2550

HW 2 - Part 2

Solutions

① (a)

Let A, B, C be 2×2 matrices.

Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $C = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$

where $a, b, c, d, e, f, g, h, i, j, k, l$ are real numbers.

We have that

$$(B+C)A = \begin{pmatrix} e+i & f+j \\ g+k & h+l \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} (e+i \quad f+j)\begin{pmatrix} a \\ c \end{pmatrix} & (e+i \quad f+j)\begin{pmatrix} b \\ d \end{pmatrix} \\ (g+k \quad h+l)\begin{pmatrix} a \\ c \end{pmatrix} & (g+k \quad h+l)\begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} (e+i)a + (f+j)c & (e+i)b + (f+j)d \\ (g+k)a + (h+l)c & (g+k)b + (h+l)d \end{pmatrix}$$

$$= \begin{pmatrix} ea + ia + fc + jc & eb + ib + fd + jd \\ ga + ka + hc + lc & gb + kb + hd + ld \end{pmatrix}$$

Also,

$$BA+CA = \begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} \bar{i} & \bar{j} \\ k & l \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} (e\ f)\begin{pmatrix} a \\ c \end{pmatrix} & (e\ f)\begin{pmatrix} b \\ d \end{pmatrix} \\ (g\ h)\begin{pmatrix} a \\ c \end{pmatrix} & (g\ h)\begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix} + \begin{pmatrix} (\bar{i}\ \bar{j})\begin{pmatrix} a \\ c \end{pmatrix} & (\bar{i}\ \bar{j})\begin{pmatrix} b \\ d \end{pmatrix} \\ (k\ l)\begin{pmatrix} a \\ c \end{pmatrix} & (k\ l)\begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} ea+fc & eb+fd \\ ga+hc & gb+hd \end{pmatrix} + \begin{pmatrix} \bar{i}a+\bar{j}c & \bar{i}b+\bar{j}d \\ ka+lc & kb+ld \end{pmatrix}$$

$$= \begin{pmatrix} ea+fc+\bar{i}a+\bar{j}c & eb+fd+\bar{i}b+\bar{j}d \\ ga+hc+ka+lc & gb+hd+kb+ld \end{pmatrix}$$

Comparing these two results we see that $(B+C)A = BA+CA$.

① (b)

Let A be a 2×2 matrix and I be the 2×2 identity matrix.

Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a,b,c,d \in \mathbb{R}$

and $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

We have that

$$IA = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} (1 \;\; 0)\begin{pmatrix} a \\ c \end{pmatrix} & (1 \;\; 0)\begin{pmatrix} b \\ d \end{pmatrix} \\ (0 \;\; 1)\begin{pmatrix} a \\ c \end{pmatrix} & (0 \;\; 1)\begin{pmatrix} b \\ d \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot a + 0\cdot c & 1\cdot b + 0\cdot d \\ 0\cdot a + 1\cdot c & 0\cdot b + 1\cdot d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

① (c)

Let A be a 2×2 matrix and O be the 2×2 zero matrix.

Then, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where $a, b, c, d \in \mathbb{R}$

and $O = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$.

Then,

$$A + O = \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} a+0 & b+0 \\ c+0 & d+0 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} = A$$

① (d)

Let A be a 2×2 matrix and α, β be real numbers.

Then, $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ where a, b, c, d are real numbers.

So,

$$(\alpha+\beta) A = (\alpha+\beta) \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$= \begin{pmatrix} (\alpha+\beta)a & (\alpha+\beta)b \\ (\alpha+\beta)c & (\alpha+\beta)d \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a + \beta a & \alpha b + \beta b \\ \alpha c + \beta c & \alpha d + \beta d \end{pmatrix}$$

$$= \begin{pmatrix} \alpha a & \alpha b \\ \alpha c & \alpha d \end{pmatrix} + \begin{pmatrix} \beta a & \beta b \\ \beta c & \beta d \end{pmatrix}$$

$$= \alpha \begin{pmatrix} a & b \\ c & d \end{pmatrix} + \beta \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \alpha A + \beta A$$

1 (e)

Let A, B, C be 2×2 matrices.

Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$, $C = \begin{pmatrix} i & j \\ k & l \end{pmatrix}$

where $a, b, c, d, e, f, g, h, i, j, k, l$ are real numbers.

Then,

$$A(BC) = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\left[\begin{pmatrix} e & f \\ g & h \end{pmatrix}\begin{pmatrix} i & j \\ k & l \end{pmatrix}\right]$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} ei+fk & ej+fl \\ gi+hk & gj+hl \end{pmatrix}$$

$$= \begin{pmatrix} (a\ \ b)\begin{pmatrix} ei+fk \\ gi+hk \end{pmatrix} & (a\ \ b)\begin{pmatrix} ej+fl \\ gj+hl \end{pmatrix} \\ (c\ \ d)\begin{pmatrix} ei+fk \\ gi+hk \end{pmatrix} & (c\ \ d)\begin{pmatrix} ej+fl \\ gj+hl \end{pmatrix} \end{pmatrix}$$

$$= \begin{pmatrix} a(ei+fk)+b(gi+hk) & a(ej+fl)+b(gj+hl) \\ c(ei+fk)+d(gi+hk) & c(ej+fl)+d(gj+hl) \end{pmatrix}$$

$$= \begin{pmatrix} aei + afk + bgi + bhk & aej + afl + bgj + bhl \\ cei + cfk + dgi + dhk & cej + cfl + dgj + dhl \end{pmatrix}$$

Also,

$$(AB)C = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} e & f \\ g & h \end{pmatrix} \right] \begin{pmatrix} i & j \\ k & l \end{pmatrix}$$

$$= \begin{pmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{pmatrix} \begin{pmatrix} i & j \\ k & l \end{pmatrix}$$

$$= \begin{pmatrix} (ae+bg)i + (af+bh)k & (ae+bg)j + (af+bh)l \\ (ce+dg)i + (cf+dh)k & (ce+dg)j + (cf+dh)l \end{pmatrix}$$

$$= \begin{pmatrix} aei + bgi + afk + bhk & aej + bgj + afl + bhl \\ cei + dgi + cfk + dhk & cej + dgj + cfl + dhl \end{pmatrix}$$

By comparing the above two results we see that $A(BC) = (AB)C$.

① (f)

Let A and B be 2×2 matrices.

Then $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ and $B = \begin{pmatrix} e & f \\ g & h \end{pmatrix}$

where a, b, c, d, e, f, g, h are real numbers.

Then,

$$(A+B)^T = \left[\begin{pmatrix} a & b \\ c & d \end{pmatrix} + \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right]^T$$

$$= \begin{pmatrix} a+e & b+f \\ c+g & d+h \end{pmatrix}^T$$

$$= \begin{pmatrix} a+e & c+g \\ b+f & d+h \end{pmatrix}$$

$$= \begin{pmatrix} a & c \\ b & d \end{pmatrix} + \begin{pmatrix} e & g \\ f & h \end{pmatrix}$$

$$= \begin{pmatrix} a & b \\ c & d \end{pmatrix}^T + \begin{pmatrix} e & f \\ g & h \end{pmatrix}^T = A^T + B^T$$

(2) (a) Let A, B, C, D be $n \times n$ matrices.

Then,

From class: $X(Y+Z) = XY + XZ$

$$(A+B)(C+D) = (A+B)C + (A+B)D$$

$$= AC + BC + AD + BD$$

From class: $(X+Y)Z = XZ + YZ$

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(2) (b) Let A, B, C, D be $n \times n$ matrices.

Then,

From class: $(X+Y)Z = XZ + YZ$

$$(\underbrace{A+B}_{X} + \underbrace{C}_{Y})\underbrace{D}_{Z} = (A+B)D + CD$$

$$= AD + BD + CD$$

From class: $(X+Y)Z = XZ + YZ$

(2)(c) Let A, B, C be $n \times n$ matrices.

Then

$\boxed{\text{From class: } (X+Y)^T = X^T + Y^T}$

$$(\underbrace{A+B}_{X}+\underbrace{C}_{Y})^T = (A+B)^T + C^T$$

$$= A^T + B^T + C^T$$

$\boxed{\text{From class: } (X+Y)^T = X^T + Y^T}$