We might be in BIO 334 next time

Another $A\vec{x} = \vec{b}$ example

Ex:

$$\begin{aligned} z + 7y - w + x &= 1 \\ y + 2w + 2x &= 0 \\ 130z \quad + w - x &= 5 \end{aligned}$$

$$A = \begin{array}{c} \begin{pmatrix} 1 & 7 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 130 & 0 & 1 & -1 \end{pmatrix} \\ \begin{array}{cccc} \uparrow & \uparrow & \uparrow & \uparrow \\ z & y & w & x \end{array} \end{array} \qquad \vec{x} = \begin{pmatrix} z \\ y \\ w \\ x \end{pmatrix} \qquad \vec{b} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

Check:

$$A\vec{x} = \vec{b}$$

$$\underbrace{\begin{pmatrix} 1 & 7 & -1 & 1 \\ 0 & 1 & 2 & 2 \\ 130 & 0 & 1 & -1 \end{pmatrix}}_{3\times4} \underbrace{\begin{pmatrix} z \\ y \\ w \\ x \end{pmatrix}}_{4\times1} = \underbrace{\begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}}_{3\times1}$$

✓

answer is 3x1

$$\begin{pmatrix} (1 \; 7 \; -1 \; 1) \cdot \begin{pmatrix} z \\ y \\ w \\ x \end{pmatrix} \\ (0 \; 1 \; 2 \; 2) \cdot \begin{pmatrix} z \\ y \\ w \\ x \end{pmatrix} \\ (130 \; 0 \; 1 \; -1) \cdot \begin{pmatrix} z \\ y \\ w \\ x \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

$$\begin{pmatrix} z + 7y - w + x \\ y + 2w + 2x \\ 130z \quad + w - x \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 5 \end{pmatrix}$$

$$\begin{aligned} z + 7y - w + x &= 1 \\ y + 2w + 2x &= 0 \\ 130z \quad + w - x &= 5 \end{aligned}$$

Same equations!

We are going learn how to solve any linear system.

First we need some definitions/tools

Def: Given a system of linear equations there are three operations that we call <u>elementary row operations</u>

1. Multiply a row/equation by a non-zero number.
2. Interchange two rows/equations
3. Add a multiple of one row/equation to a different row/equation.

Ex: (Multiply a row/equation by a non-zero number)

system viewpiont

$$\begin{cases} 3x + y - z = 1 \\ \frac{1}{2}x + 5y = 2 \\ y + z = 10 \end{cases} \xrightarrow{2 \cdot R_2 \to R_2} \begin{cases} 3x + y - z = 1 \\ x + 10y = 4 \\ y + z = 10 \end{cases}$$

augmented matrix viewpoint

$$\left(\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ \frac{1}{2} & 5 & 0 & 2 \\ 0 & 1 & 1 & 10 \end{array}\right) \xrightarrow{2R_2 \to R_2} \left(\begin{array}{ccc|c} 3 & 1 & -1 & 1 \\ 2 & 10 & 0 & 4 \\ 0 & 1 & 1 & 10 \end{array}\right)$$

Ex: (Interchanging two rows)

system viewpoint

$$\begin{cases} x + y = 3 \\ 2x - y = 7 \\ x - 10y = 1 \end{cases} \xrightarrow{R_2 \leftrightarrow R_3} \begin{cases} x + y = 3 \\ x - 10y = 1 \\ 2x - y = 7 \end{cases}$$

augmented matrix viewpoint

$$\left(\begin{array}{cc|c} 1 & 1 & 3 \\ 2 & -1 & 7 \\ 1 & -10 & 1 \end{array}\right) \xrightarrow{R_2 \leftrightarrow R_3} \left(\begin{array}{cc|c} 1 & 1 & 3 \\ 1 & -10 & 1 \\ 2 & -1 & 7 \end{array}\right)$$

Ex: (Add a multiple one one row to a different row)

system viewpoint

$$\boxed{\begin{aligned} x+y&=3\\ 2x-y&=7\\ x-10y&=1 \end{aligned}} \xrightarrow{-2R_1+R_2\to R_2} \boxed{\begin{aligned} x+y&=3\\ -3y&=1\\ x-10y&=1 \end{aligned}}$$

$$\begin{aligned} -2x-2y&=-6 && \leftarrow -2R_1\\ +(2x-y&=7) && \leftarrow +R_2\\ \hline 0x-3y&=1 && \leftarrow \boxed{\text{new } R_2} \end{aligned}$$

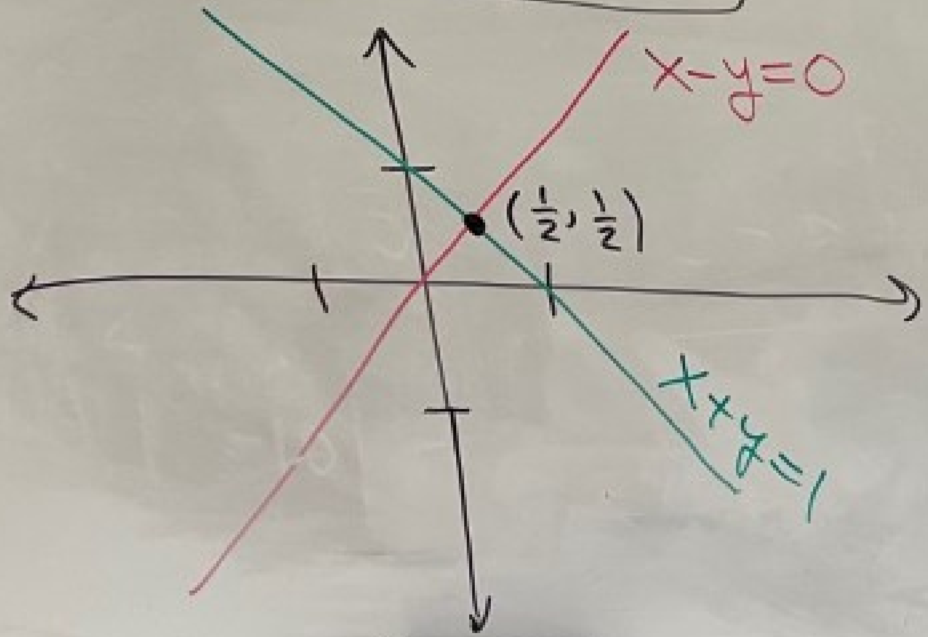
matrix viewpoint

$$\left(\begin{array}{cc|c} 1 & 1 & 3\\ 2 & -1 & 7\\ 1 & -10 & 1 \end{array}\right) \xrightarrow{-2R_1+R_2\to R_2} \left(\begin{array}{cc|c} 1 & 1 & 3\\ 0 & -3 & 1\\ 1 & -10 & 1 \end{array}\right)$$

$$\begin{aligned} &\left(\begin{array}{cc|c} -2 & -2 & -6 \end{array}\right) && \leftarrow -2R_1\\ +&\left(\begin{array}{cc|c} 2 & -1 & 7 \end{array}\right) && \leftarrow +R_2\\ \hline &\left(\begin{array}{cc|c} 0 & -3 & 1 \end{array}\right) && \leftarrow \text{new } R_2 \end{aligned}$$

Theorem: Applying an elementary row operation to a system of linear equations does not change the solution space of the system

Ex: $\boxed{\begin{aligned} x+y&=1 \\ x-y&=0 \end{aligned}}$ $(*)$

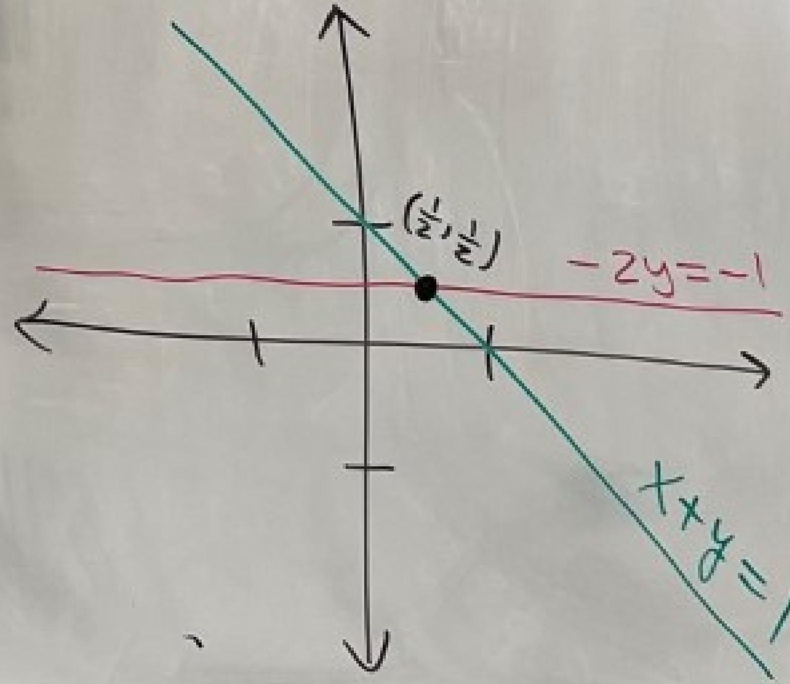


Solution space of $(*)$ is $\{(\frac{1}{2},\frac{1}{2})\}$

Let's apply an elementary row operation

$$\boxed{\begin{aligned} x+y&=1 \\ x-y&=0 \end{aligned}} \xrightarrow{-R_1+R_2 \to R_2} \boxed{\begin{aligned} x+y&=1 \\ -2y&=-1 \end{aligned}}$$

$$\begin{array}{rl} -x-y=-1 & \leftarrow -R_1 \\ +\ x-y=0 & \leftarrow +R_2 \\ \hline -2y=-1 & \leftarrow \text{new } R_2 \end{array}$$



We see that the solution space of the new system is the same as the original, ie it is still $\{(\frac{1}{2},\frac{1}{2})\}$

Def: If a row of a matrix does not consist entirely of zeros then the <u>leading entry</u> in that row is the first non-zero entry when scanning from left to right.

Ex:

$$\begin{array}{cl} \begin{pmatrix} 10 & 0 & 3 & -2 \\ 0 & 0 & 1 & 7 \\ 1 & 0 & 2 & 3 \\ 0 & \frac{1}{2} & \pi & 71 \\ 0 & 0 & 0 & 0 \end{pmatrix} & \begin{array}{l} \leftarrow \text{row } 1 \\ \leftarrow \text{row } 2 \\ \leftarrow \text{row } 3 \\ \leftarrow \text{row } 4 \\ \leftarrow \text{row } 5 \end{array} \end{array}$$

10 is the leading entry in row 1
1 is the leading entry in row 2
1 is the leading entry in row 3
$1/2$ is the leading entry in row 4
row 5 has no leading entry