Def:

A <u>system of m linear equations with n unknowns $x_1, x_2, \ldots, x_n$</u> is m linear equations of the form

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ &\vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}}$$

Where the a_{ij} and b_i are constants.

The <u>augmented matrix</u> for this system is

$$\left(\begin{array}{cccc|c} a_{11} & a_{12} & \cdots & a_{1n} & b_1 \\ a_{21} & a_{22} & \cdots & a_{2n} & b_2 \\ \vdots & \vdots & & \vdots & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} & b_m \end{array}\right)$$

The <u>solution space</u> of the system consists of all the $(x_1, x_2, \dots, x_n)$ that simultaneously solve all m equations.

That is, the common solutions to all m equations.

<u>Ex:</u>

$$\left.\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned}\right] \quad \begin{aligned} m &= 2 \text{ lin. eqs} \\ n &= 2 \text{ unknowns} \end{aligned}$$

$$\boxed{\text{Augmented matrix}} \rightarrow \left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 5 & 6 \end{array}\right)$$

$y = -\frac{1}{2}x + \frac{3}{2}$

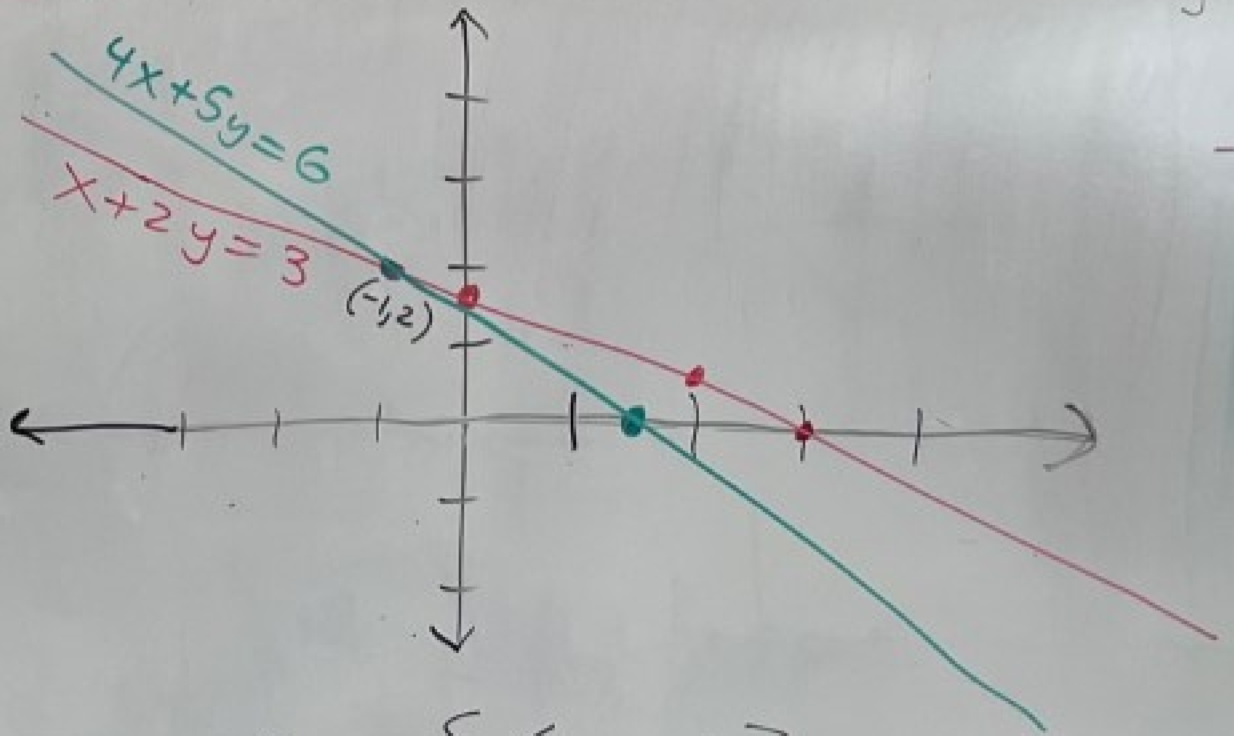
$y = -\frac{4}{5}x + \frac{3}{2}$,

$-\frac{1}{2}(2) + \frac{3}{2}$

$-1 + \frac{3}{2}$

$= 0.5$

$\boxed{\text{Solution space}}$

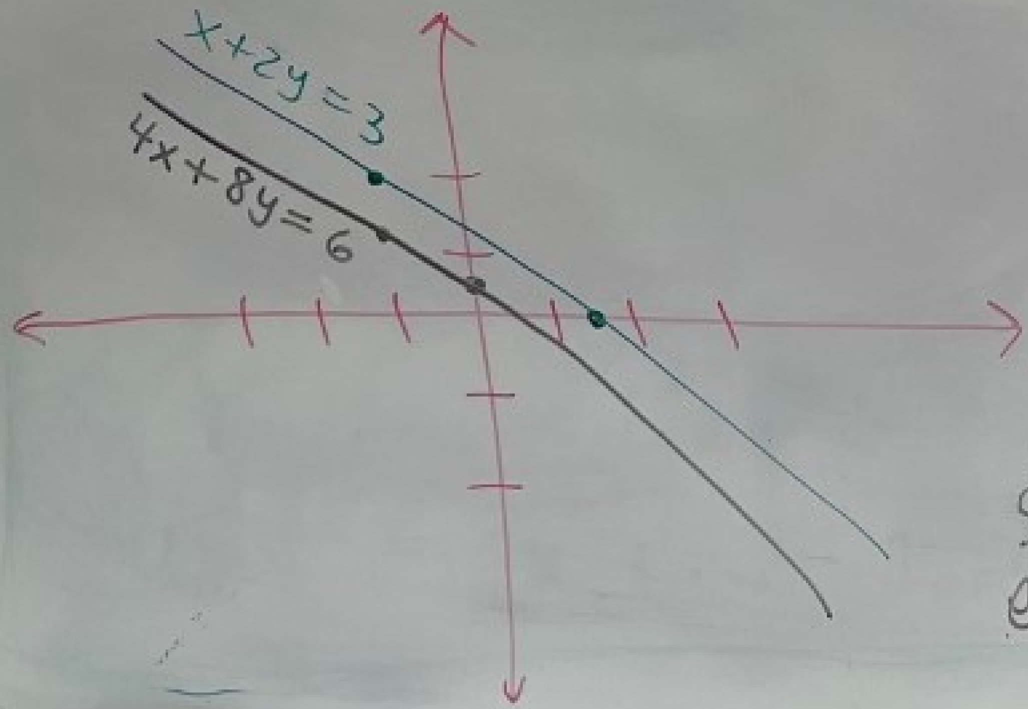


Solution space is $\{(-1,2)\}$

Ex: Consider the system

$$\begin{aligned} x+2y &= 3 \\ 4x+8y &= 6 \end{aligned}$$

Augmented matrix $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 6 \end{array}\right)$

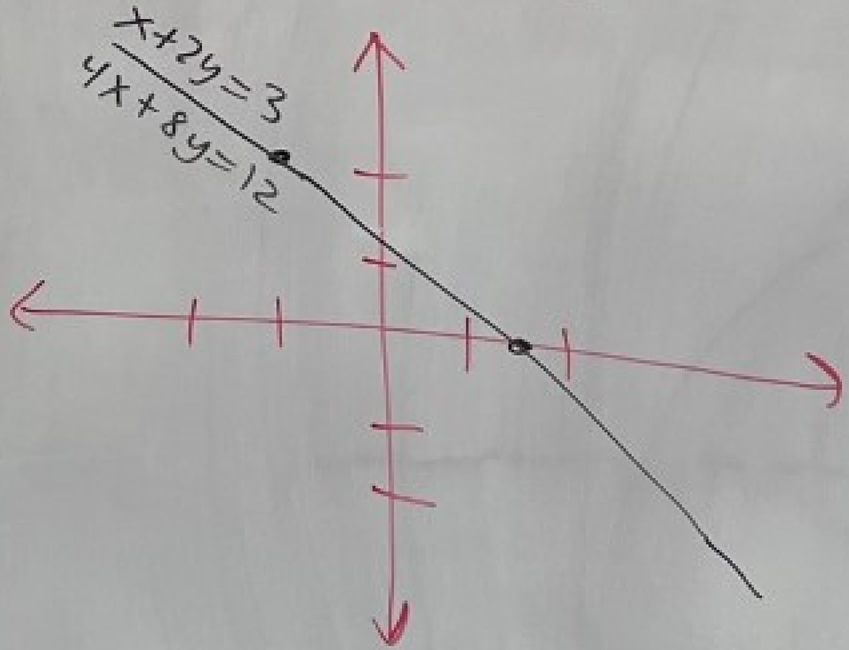


Solution Space

No common solutions, because parallel lines. Solution space is empty set $\emptyset$

Ex: $\left.\begin{aligned} x+2y &= 3 \\ 4x+8y &= 12 \end{aligned}\right]$ same line twice

Augmented matrix $\left(\begin{array}{cc|c} 1 & 2 & 3 \\ 4 & 8 & 12 \end{array}\right)$



Solution space is entire line. Infinitely many solutions.

Ex:

$$\left.\begin{aligned} x + y + 2z &= 9 \\ 2x \quad\quad - 3z &= 1 \\ -x + 6y - 5z &= 0 \end{aligned}\right]$$

$m = 3$ eqns

$n = 3$ unknowns

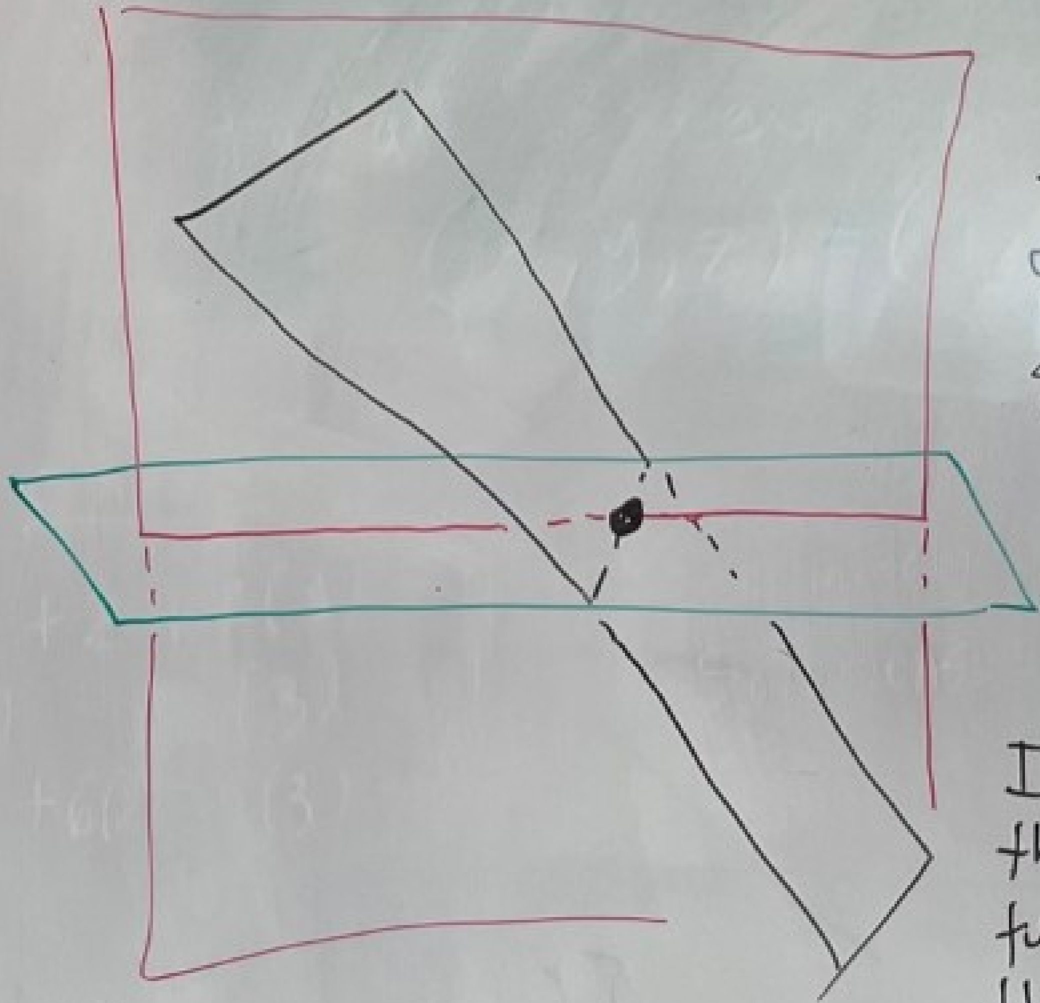
These are 3 planes in 3d space

Augmented matrix

$$\left(\begin{array}{ccc|c} 1 & 1 & 2 & 9 \\ 2 & 0 & -3 & 1 \\ -1 & 6 & -5 & 0 \end{array}\right)$$

$$\begin{array}{ccc} \uparrow & \uparrow & \uparrow \\ x & y & z \end{array}$$

Later we will find the solution space.



You could have something like this with one common solution.

I think this example turns out this way.

Ex:

$$
\begin{aligned}
10x - 3y \phantom{{}+y - z} &+ w = 1 \\
3x + y - z &+ 2w = 0 \\
10x \phantom{{} - 3y - z} &+ w = 7
\end{aligned}
$$

$m = 3$ eqns, $n = 4$ unknowns

$$
\begin{array}{c}
\left(\begin{array}{cccc|c}
10 & -3 & 0 & 1 & 1 \\
3 & 1 & -1 & 2 & 0 \\
10 & 0 & 0 & 1 & 7
\end{array}\right) \\
\begin{array}{cccc}
x & y & z & w
\end{array}
\end{array}
$$

augmented matrix

There's another way to represent a system of linear equations.

Given the system

$$\boxed{\begin{aligned} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n &= b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n &= b_2 \\ \vdots \qquad\qquad \vdots \qquad\qquad & \quad \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n &= b_m \end{aligned}}$$

Let

$$A = \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix}, \quad \vec{x} = \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}, \quad \vec{b} = \begin{pmatrix} b_1 \\ b_2 \\ \vdots \\ b_m \end{pmatrix}$$

The system can be represented by the matrix equation

$$\underbrace{A\vec{x}}_{\text{matrix multiplication}} = \vec{b}$$

Ex:

$$\boxed{\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned}}$$

$$\boxed{\begin{aligned} A &= \begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix} \\ \vec{x} &= \begin{pmatrix} x \\ y \end{pmatrix} \\ \vec{b} &= \begin{pmatrix} 3 \\ 6 \end{pmatrix} \end{aligned}}$$

$$A\vec{x} = \vec{b}$$

$$\underbrace{\underbrace{\begin{pmatrix} 1 & 2 \\ 4 & 5 \end{pmatrix}}_{2\times 2}\underbrace{\begin{pmatrix} x \\ y \end{pmatrix}}_{2\times 1}}_{\text{answer} = 2\times 1} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} (1 \;\; 2)\cdot\begin{pmatrix} x \\ y \end{pmatrix} \\ (4 \;\; 5)\cdot\begin{pmatrix} x \\ y \end{pmatrix} \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix}$$

$$\begin{pmatrix} x + 2y \\ 4x + 5y \end{pmatrix} = \begin{pmatrix} 3 \\ 6 \end{pmatrix} \leftarrow \boxed{\begin{aligned} x + 2y &= 3 \\ 4x + 5y &= 6 \end{aligned}}$$