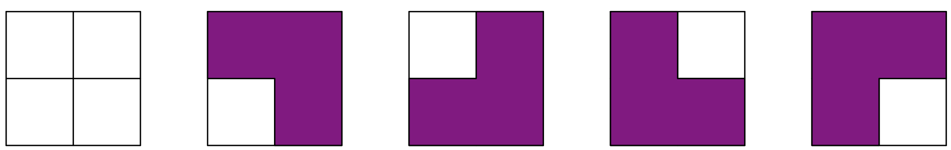
Tiling with Ls and Squares

Silvia Heubach

Department of Mathematics
California State University Los Angeles



joint work with
Phyllis Chinn and Ralph Grimaldi

Many authors have looked at all kinds of tilings problems ... a **tiny** sampling:

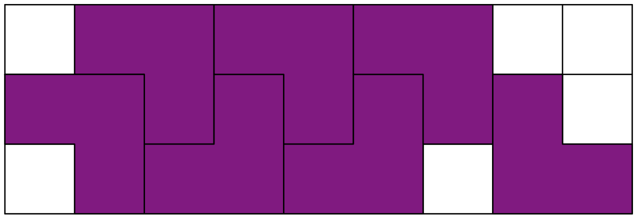
- E.O. Hare and P.Z. Chinn, Tiling with Cuisenaire rods, *Applications of Fibonacci numbers* **6** (1994), 165–171.
- S. Heubach, Tiling an m-by-n area with squares of size up to k-by-k with $m \leq 5$, *Congressus Numerantium* **140** (1999), 43-64.
- R. Hochberg and M. Reid, Tiling with notched cubes, *Discrete Math.* **214** (2000), 255–261.
- A. T. Benjamin and J.J. Quinn, *Proofs that Really Count*, MAA 2003

Things to come ...

- We look at $2{\times}n$ and $3{\times}n$ boards
- Count number of tilings $T_{m,n}$ of an $m{\times}n$ board
- Count number of Ls, $T^{L}_{m,n}$, and squares, $T^{S}_{m,n}$, in all such tilings
- $2{\times}n \rightsquigarrow$ explicit results, "easy case"
- $3{\times}n \rightsquigarrow$ different techniques needed

Definitions and basic ideas

- The generating function for a sequence $\{a_{m,n}\}_{n=0}^{\infty}$ is defined as $G_{a_m}(x) = \sum_{n=0}^{\infty} a_{m,n} x^n$.
- A basic block is a tiling that cannot be split vertically into smaller rectangular tilings. We denote the number of basic blocks of size $m \times k$ by $B_{m,k}$
- Any tiling is composed of a basic block followed by a smaller tiling.



$$\begin{aligned} T_{m,n} &= \sum_{k=1}^{n} B_{m,k} \cdot T_{m,n-k} \quad \text{for} \quad m,n \geq 1 \quad (*) \\ T_{m,0} &= 1 \quad \text{for} \quad m \geq 1 \end{aligned}$$

Recursion for $T_{m,n}$ is a convolution $\Rightarrow$ generating functions multiply. Multiplying (*) by x^n and summing over $n \geq 1$ we obtain

$$G_{T_m}(x) - 1 = G_{B_m}(x) G_{T_m}(x)$$

$$\Rightarrow G_{T_m}(x) = \frac{1}{1 - G_{B_m}(x)}$$

Connection between the various quantities

- Looking at total area gives

$$m \cdot n \cdot T_{m,n} = 3T^L_{m,n} + T^S_{m,n}$$

$\Rightarrow$ need to count only squares or Ls

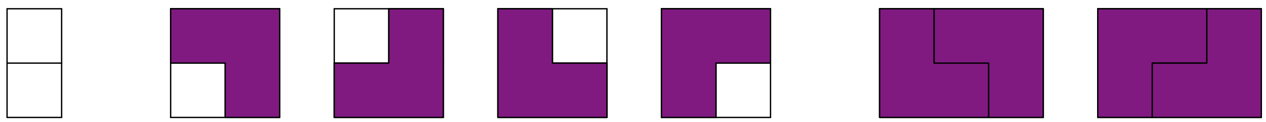
- Counting Squares

$$T^S_{m,n} = \underbrace{\sum_{k=1}^{n} B_{m,k} \cdot T^S_{m,n-k}}_{\text{squares from tilings}} + \underbrace{\sum_{k=1}^{n} B^S_{m,k} \cdot T_{m,n-k}}_{\text{squares from basic blocks}} \qquad (**)$$

- Corresponding formula holds for Ls

Tiling 2×n boards

- Basic blocks of size $2\times k$



- Recursion $T_{2,n} = T_{2,n-1} + 4 \cdot T_{2,n-2} + 2 \cdot T_{2,n-3}$
- Characteristic equation $x^3 - x^2 - 4x - 2$ has roots -1, $1+\sqrt{3}$, and $1-\sqrt{3}$

Theorem: The number of tilings of size $2\times n$ with L-shaped tiles and squares for $n \geq 0$ is given by

$$T_{2,n} = (-1)^n + (1/\sqrt{3})(1+\sqrt{3})^n + (-1/\sqrt{3})(1-\sqrt{3})^n,$$

with generating function

$$G_{T_{2,n}}(x) = 1/(1-x-4x^2-2x^3).$$

The values for $T_{2,n}$ for $n = 0, \ldots, 20$ are given by {1, 1, 5, 11, 33, 87, 241, 655, 1793, 4895, 13377, 36543, 99841, 272767, 745217, 2035967, 5562369, 15196671, 41518081, 113429503, 309895169}

Counting Ls and Squares

We give formulas for $T^L_{2,n}$ and $T^S_{2,n}$ in three forms: combinatorial, explicit and generating function.

Combinatorial set-up:

- Tiling consists of sequence of basic blocks
- $s = \#$ of basic blocks of width 1, $d = \#$ of basic blocks of width 2, $t = \#$ of basic blocks of width 3.
- $s + 2d + 3t = n \Rightarrow$

  $\#$ of basic blocks $\ell = s + d + t = n - d - 2t$

Theorem: The number of squares and L-shaped tiles for tilings of the $2\times n$ board are given in combinatorial form by

$$T^S_{2,n} = \sum_{t=0}^{n/3} \sum_{d=0}^{(n-3t)/2} \binom{\ell}{t}\binom{\ell-t}{d}(2s+d)4^d 2^t$$

$$T^L_{2,n} = \sum_{t=0}^{n/3} \sum_{d=0}^{(n-3t)/2} \binom{\ell}{t}\binom{\ell-t}{d}(d+2t)4^d 2^t,$$

where $s = n - 3t - 2d$.

Explicitly:

$$
\begin{aligned}
T_{2,n}^{S} \quad = \quad & (2n-12)(-1)^{n} \\
& + \frac{2}{3}((9-5\sqrt{3})(1+\sqrt{3})^{n} + (9+5\sqrt{3})(1-\sqrt{3})^{n}) \\
& + \frac{n}{\sqrt{3}}((\sqrt{3}-1)(1+\sqrt{3})^{n} + (\sqrt{3}+1)(1-\sqrt{3})^{n}).
\end{aligned}
$$

and

$$
\begin{aligned}
T_{2,n}^{L} \quad = \quad & 4(-1)^{n} - \frac{2}{9}((9-5\sqrt{3})(1+\sqrt{3})^{n} + (9+5\sqrt{3})(1-\sqrt{3})^{n} \\
& - \frac{n}{3}((1-\sqrt{3})(1+\sqrt{3})^{n} + (1+\sqrt{3})(1-\sqrt{3})^{n}).
\end{aligned}
$$

The respective generating functions are

$$G_{T_2^S}(x) = \frac{2x(1+2x)}{(1+x)^2(1-2x-2x^2)^2}$$

and

$$G_{T_2^L}(x) = \frac{4x^2}{(1+x)(1-2x-2x^2)^2}.$$

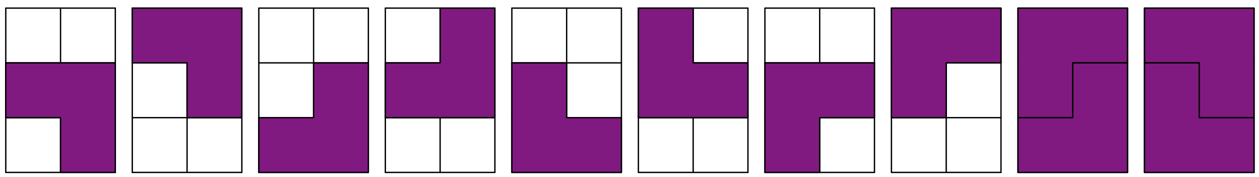
Proof (Outline): From (**) we get (for Ls)

$$T^{L}_{2,n} = T^{L}_{2,n-1} + 4 \cdot T^{L}_{2,n-2} + 2 \cdot T^{L}_{2,n-3} + 4 \cdot T_{2,n-2} + 4 \cdot T_{2,n-3}$$

- Homogeneous part $\leftrightarrow T_{2,n}$; for inhomogeneous part substitute result for $T_{2,n} \Rightarrow$ linear comb. of $(1+\sqrt{3})$ and $(1-\sqrt{3})$
- Solve general solution for $T^{L}_{2,n}$ using initial conditions, then use area formula for $T^{S}_{2,n}$
- Generating functions easy - (**) consists of two convolutions, and $G_{B^S_2}(x) = 2x + 4x^2$ and $G_{B^L_2}(x) = 4x^2 + 4x^3$.

Tiling $3\times n$ boards

- One basic block of size $3\times1 \leftrightarrow$ all squares
- Eight basic blocks of size 3×2



- Basic blocks of bigger size ????? $\rightsquigarrow$ recursive algorithm to create basic blocks of size $3\times(k+1)$ from those of size $3\times k$
- Look at the last "column" and refer to each 1×1 area of the tiling as a cell

Basic Block Creation (BBC) Algorithm:

- *Type I*: The last column has one cell covered by a square and two (adjacent) cells covered by an L.
- *Type II*: The last column has two non-adjacent cells covered with squares.
- *Type III*: The last column has two adjacent cells covered with squares.

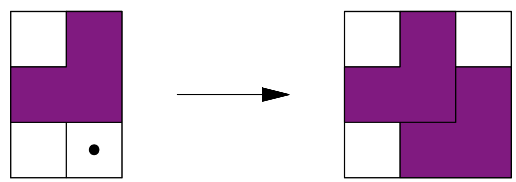
Other possibilities for last column:

- All cells are covered with squares (only for $k = 1$)
- All cells are covered with L tiles (only for $k = 2$)

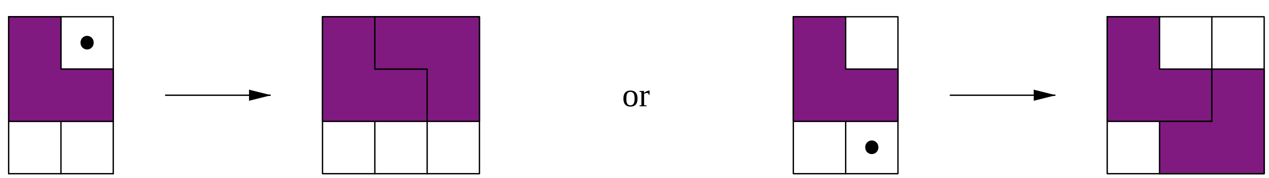
Neither of these basic blocks can be extended.

Extensions

- Each Type I extension produces one Type I basic block

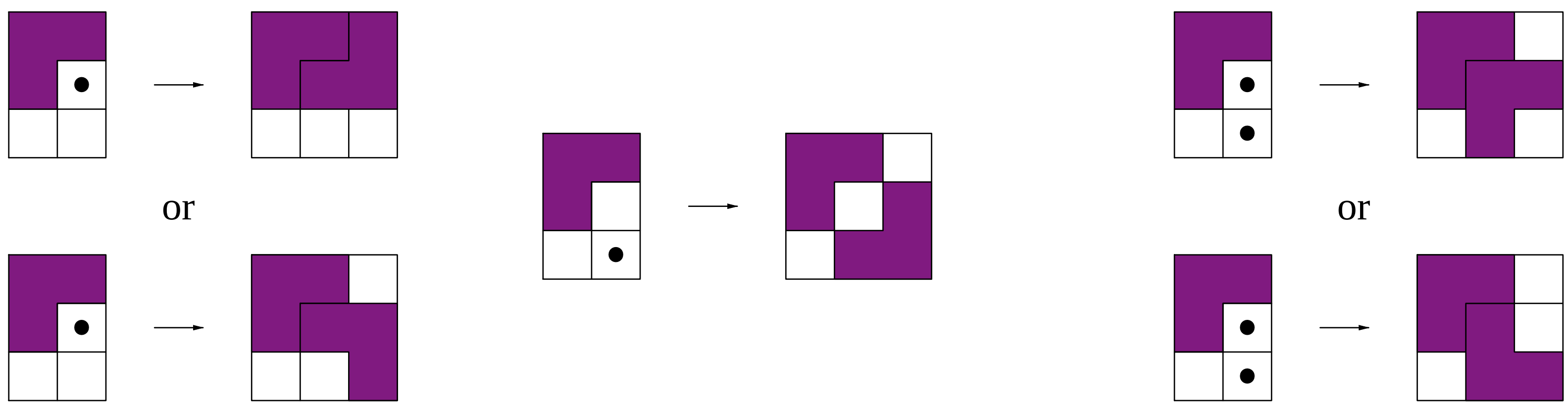


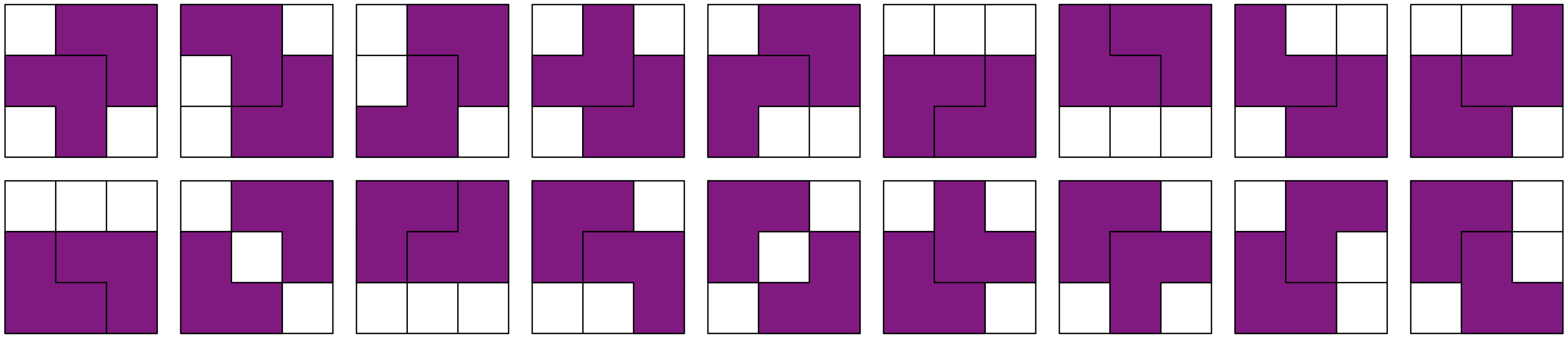
- Each Type II extension produces two Type I basic blocks

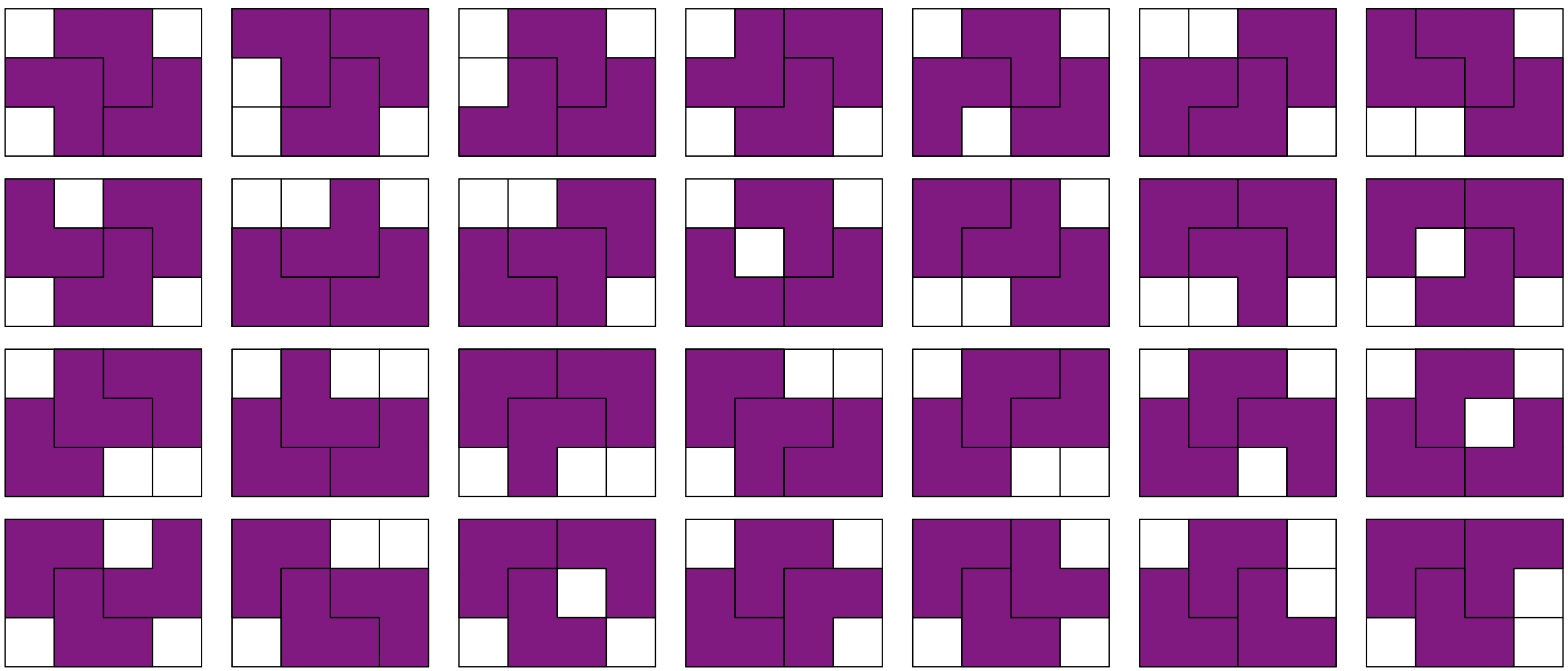


Type III Extensions

Each Type III extension produces three Type I, one Type II, and one Type III basic blocks.







Theorem: The number of basic blocks is given by

$$B_{3,1} = 1, B_{3,2} = 10, \text{ and } B_{3,k} = 10k - 12 \quad \text{for } k \geq 3.$$

The generating functions for the number of basic blocks and the number of tilings are given by

$$G_{B_3}(x) = \frac{x + 8x^2 - x^3 + 2x^4}{(1-x)^2}$$

and

$$G_{T_3}(x) = \frac{(1-x)^2}{1 - 3x - 7x^2 + x^3 - 2x^4}.$$

The values for $\{T_{3,n}\}_{n=0}^{15} = \{1, 1, 11, 39, 195, 849, 3895, 17511, 79339, 358397, 1620843, 7326991, 33127155, 149766353, 677103839, 3061202815\}$.

Proof (Outline):

- For $k \geq 3$, $B_{3,k} = b_{k,I} + b_{k,II} + b_{k,III}$
- Initial conditions $b_{2,I} = 4$, $b_{2,II} = 2$, and $b_{2,III} = 2$
- Extension algorithm gives recursions $\leadsto$ solve

$$\begin{aligned} b_{k+1,III} &= b_{k,III} (= \ldots = b_{2,III} = 2) \\ b_{k+1,II} &= b_{k,III} = 2 \\ b_{k+1,I} &= b_{k,I} + 2b_{k,II} + 3b_{k,III} = b_{k,I} + 10. \end{aligned}$$

- Gf for $B_{3,k}$ from explicit formula; gf for $T_{3,k}$ from convolution

Theorem: The generating functions for the number of squares and L-shaped tiles in all tilings of the $3\times n$ board are given by

$$G_{T_3^S}(x) = \frac{3x(1-x)^2(1+6x+3x^2)}{(1-3x-7x^2+x^3-2x^4)^2}$$

and

$$G_{T_3^L}(x) = \frac{4x^2(3-3x+3x^2-4x^3+x^4)}{(1-3x-7x^2+x^3-2x^4)^2}.$$

Proof (Outline):

- Note that each basic block (for $k \geq 3$) has exactly 3 squares - proof follows from the extensions
- Compute gf for $B^S_{3,n}$ from gf of $B_{3,n}$ with adjustments for initial discrepancy
- Gf for $T^S_{3,n}$ from

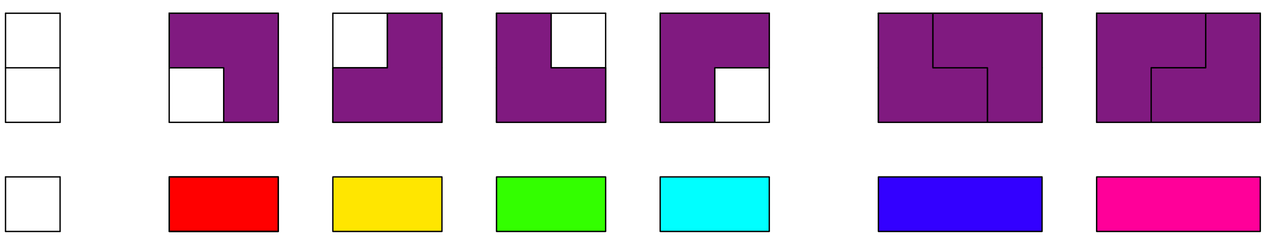
$$T^S_{m,n} = \underbrace{\sum_{k=1}^{n} B_{m,k} \cdot T^S_{m,n-k}}_{\text{squares from tilings}} + \underbrace{\sum_{k=1}^{n} B^S_{m,k} \cdot T_{m,n-k}}_{\text{squares from basic blocks}} \qquad (**)$$

- Gf for $T^L_{3,n}$ from area formula

Connection to colored $1 \times n$ boards

- Any finite recursion with the proper initial condition can be interpreted as counting the tilings of a $1 \times n$ board with colored tiles of size $1 \times k$
- Coefficients in the recursion indicate the number of tiles of the respective size
- Each basic block of a given width is mapped to a different color

$$T_{2,n} = T_{2,n-1} + 4 \cdot T_{2,n-2} + 2 \cdot T_{2,n-3}$$



Future Research

- Case $m > 3$??? $\rightsquigarrow$ there are more types for the extension algorithm
  - Is there a general structure for the types?
  - Is the a general structure for the types and number of each type that result from extension of a given type?
- Tiling $2\times n$ and $3\times n$ bracelets

Thanks for Listening

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