Edge-Nim on a Tetrahedron

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Definition of Edge-Nim

- Tokens are placed on the edges of a graph
- Two players alternate
- A move consists of
  1. ...choosing a vertex;
  2. ...removing at least one token from one or more edges incident to the chosen vertex

As usual in combinatorial games, the first player who cannot move loses. Or, equivalently, the player who removes the last token wins.
Example of a Move:
History of the Game

• At the 2013 Boca Raton conference, Breeann gave a talk on this game

• She had used it to introduce prospective teachers to research via graph theory problems

• She gave results on $C_3$ and $C_4$, and indicated that Edge-Nim on a Tetrahedron was open

• Edge-Nim on $C_3$ and $C_4$ are equivalent to Circular Nim $C(3,2)$ and $C(4,2)$
We are interested in a winning strategy. We will determine the losing positions (P-positions), those from which a player will be guaranteed to lose.

This needs the digital sum of integers (denoted by $\oplus$).

To compute, say $12 \oplus 17 \oplus 5$:

1. First express them in base 2:
   
   $12 = 01100_2 \quad 17 = 10001_2 \quad 5 = 00101_2$

2. Add them without «carrying over»:
   
   $01100_2 \oplus 10001_2 \oplus 00101_2 = 11000_2$

Here is how one will identify the edges (in black) and the vertices (in green) of our tetrahedron:

Vertices labeled in alphabetical order.
**Special Case**: If $a = 0$, it is like if that edge was not there anymore, as no move can be made on it. So flip down the edges $b$ and $c$ for convenience.
Partial result for the game when one of the edges is empty

The set $L$ of losing positions for the case $a = 0$ consists of all tetrahedra for which

1. $b + C = B + c$
2. $b \leq c$
3. $A \oplus b \oplus B = 0$

...so, red edges add to the same as the blue edges, and purple edges have digital sum 0
Example: This position belongs to L:

9 + 2 = 4 + 7 = 11

4 = 100₂
6 = 110₂
2 = 010₂
000₂
Here we show that Condition 2 \((b \leq c)\) is important and not just an expression of convenience to draw the pictures. Indeed, if one drops this condition (which is equivalent to flipping the path along the (missing vertical) edge \(a\):

...and one would have a move from \(L\) to \(L\), which is forbidden. So Condition 2 is important!
Proof Method:

Once one has a guess $S$ for the set of losing positions $L$, one must show that:

1. From a position in $S$, one cannot play to another position in $S$.
2. From a position outside $S$, one can always play to a position in $S$.

Then conclude $S = L$, the set of losing positions, and the game is solved...
Part 1:

- Play on either \( \text{AbC} \) or \( \text{ABc} \) can only be play on \( A \) (otherwise equality of sums is destroyed).
- Play on either \( \text{aBC} \) or \( \text{abc} \) must reduce both edges by the same amount to maintain equality of sums.
- In both cases, exactly one of the edges on the “digital” path is modified, which results in making the digital sum non-zero

\[ \Rightarrow \text{move from position in } S \text{ is to position not in } S. \]
Part 2: The proof is tedious. One needs to

- decompose the set $S$ into subsets with conditions like $b \leq c$, $A \leq b \oplus B$, etc. and make sure that all positions are covered
- determine how to play in each case...no obvious insights.
Let's have some fun:

How should one play here?
Answer:

Thanks!