A Generalization of the Nim and Wythoff games

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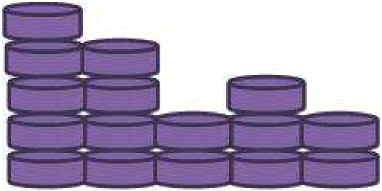
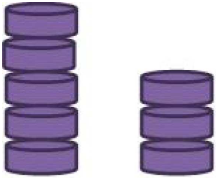
March 10, 2011
42nd CGTC Conference, Boca Raton, FL

Nim and Wythoff

- **Nim**: Select one of the n stacks, take at least one token



- **Wythoff**: Take any number of tokens from **one** stack OR select the **same** number of tokens from both stacks



Generalization of Wythoff to n stacks

Wythoff: Take any number of tokens from **one** stack OR select the **same** number of tokens from both stacks

Generalization: Take any number of tokens from **one** stack OR

- take the **same** number of tokens from ALL stacks
- take the **same** number of tokens from any TWO stacks
- take the **same** number of tokens from any non-empty SUBSET of stacks

Generalized Wythoff on n stacks

Let $B \subseteq \mathcal{P}(\{1, 2, 3, \ldots, n\})$ with the following conditions:

1. $\varnothing \notin B$
2. $\{i\} \in B$ for $i = 1, \ldots, n$.

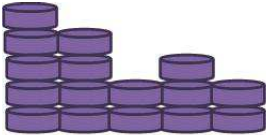
A legal move in generalized Wythoff $\mathcal{GW}_n(B)$ on n stacks induced by B consists of:

- Choose a set $A \in B$
- Remove the same number of tokens from each stack whose index is in A

The first player who cannot move loses.

Examples

- **Nim**: Select one of the n stacks, take at least one token



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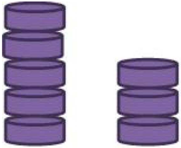

Examples

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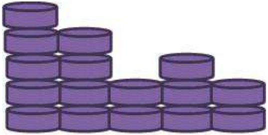
  $B = \{\{1\}, \{2\}, \ldots, \{n\}\}$



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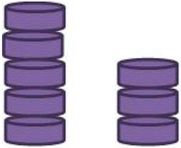

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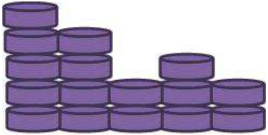

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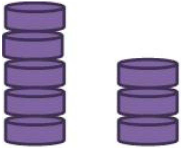

$B = \{\{1\}, \{2\}, \{1, 2\}\}$

Goal

- Generalized Wythoff is a two-player impartial game
- All positions (configurations of stack heights) are either winning or losing

Goal: Determine the set of losing positions

Smaller Goal: Say something about the structure of the losing positions

Results for Wythoff

Let $\Phi = \frac{1+\sqrt{5}}{2}$. Then the set of losing positions is given by

$$\mathcal{L} = \{(\lfloor n \cdot \Phi \rfloor, \lfloor n \cdot \Phi \rfloor + n) | n \geq 0\}$$

They can be created recursively as follows:

- For a_n, find he smallest positive integer not yet used for a_i and b_i, $i < n$.
- $b_n = a_n + n$. Repeat...

n	0	1	2	3	4	5
a_n						
b_n						

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n	0	1	2	3	4	5
a_n	0	1				
b_n	0	2				

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n	0	1	2	3	4	5
a_n	0	1	3	4	6	8
b_n	0	2	5	7	10	13

Theorem

For the game of Wythoff, for any given position (a, b), there is exactly one a losing position of the form (a, y), (x, b), $(z, z + |b - a|)$ for some $x \geq 0$, $y \geq 0$, and $z \geq 0$.

This structural result can be visualized as follows:

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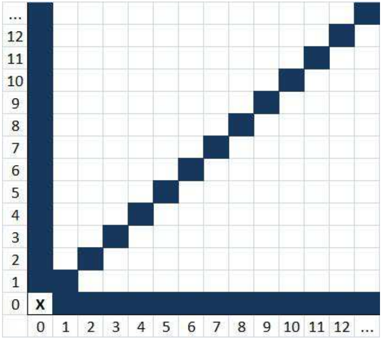

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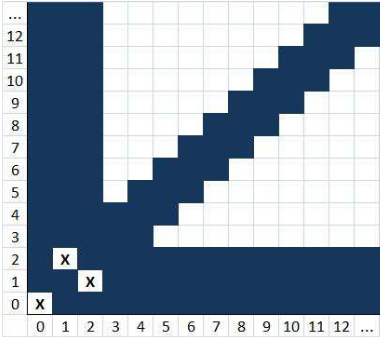

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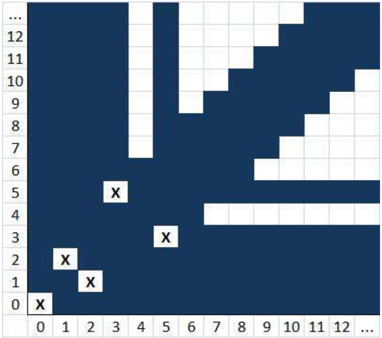

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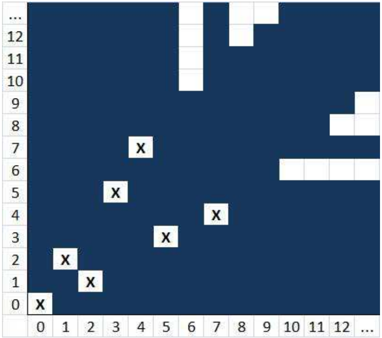

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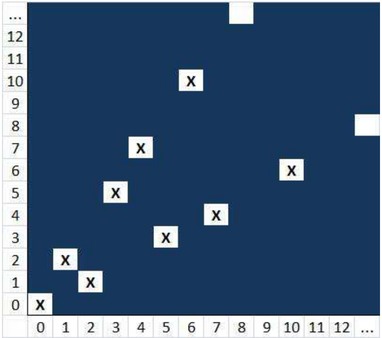

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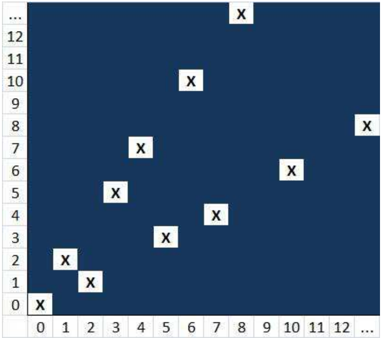

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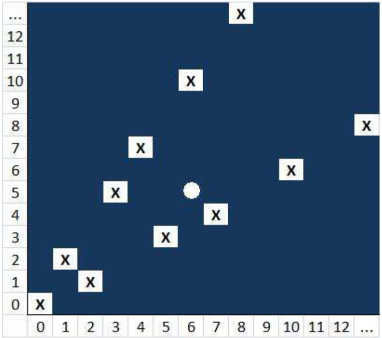
This structural result can be visualized as follows: $(a,b) = (6,5)$



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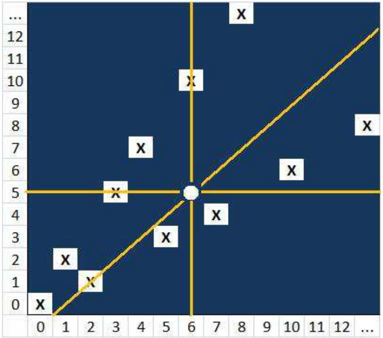

Losing positions: $(6,10)$, $(3,5)$, and $(2,1)$.

$\overrightarrow{e}_i = i^{\text{th}}$ unit vector; $\overrightarrow{e}_A = \sum_{i \in A} \overrightarrow{e}_i$

Conjecture

In the game of generalized Wythoff $\mathcal{GW}_n(B)$, for any position $\overrightarrow{p} = (p_1, p_2, \ldots, p_n)$ and any $A = \{i_1, i_2, \ldots, i_k\} \subseteq B$, there is a unique losing position of the form $\overrightarrow{p} + m \cdot \overrightarrow{e}_A$, where $m \geq -\min_{i \in A}\{p_i\}$.

Theorem

The conjecture is true for $|A| \leq 2$, that is, if play is either on a single stack or any pair of two stacks.

Example

$\mathcal{GW}_3(\{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\})$ - three stacks, with play on either a single or a pair of stacks. $\overrightarrow{p} = (11,17,20)$

A	$\tilde{p}$				
$\{1\}$	$(26,17,20)$	$=$	$(11,17,20)$	$+$	$15\cdot(1,0,0)$
$\{2\}$	$(11,31,20)$	$=$	$(11,17,20)$	$+$	$14\cdot(0,1,0)$
$\{3\}$	$(11,17,36)$	$=$	$(11,17,20)$	$+$	$16\cdot(0,0,1)$
$\{1,2\}$	$(19,25,20)$	$=$	$(11,17,20)$	$+$	$8\cdot(1,1,0)$
$\{1,3\}$	$(1,17,10)$	$=$	$(11,17,20)$	$-$	$10\cdot(1,0,1)$
$\{2,3\}$	$(11,35,38)$	$=$	$(11,17,20)$	$+$	$18\cdot(0,1,1)$

Example

$B_1 = \{\{1\},\{2\},\{3\},\{1,2\},\{1,3\},\{2,3\}\}$); $B_2 = B_1 \cup \{1,2,3\}$
$\overrightarrow{p} = (11,17,20)$

A	$\tilde{p}_1$	$\tilde{p}_2$
$\{1\}$	$(26,17,20)$	$(40,17,20)$
$\{2\}$	$(11,31,20)$	$(11,1,20)$
$\{3\}$	$(11,17,36)$	$(11,17,27)$
$\{1,2\}$	$(19,25,20)$	$(7,13,20)$
$\{1,3\}$	$(1,17,10)$	$(8,17,17)$
$\{2,3\}$	$(11,35,38)$	$(11,12,15)$
$\{1,2,3\}$	—	$(15,21,24)$

Proof Outline.

- For play on one stack, assuming no such position exists leads to contradiction (legal move from losing position to losing position) as there are only finitely many moves.
- For play on a pair of stacks, a somewhat different argument is needed that does not generalize to three or more stacks.

□

Thank You!