HOW TO WIN IN
SLOW EXACT $k$-NIM

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Slow Exact $k$-NIM $\text{SN}(n, k)$

- Play on $n$ stacks of tokens
- Move consists of
  - Picking exactly $k$ of the stacks
  - Removing one token from each of the selected stacks
- Last person to make a move wins
Known Results

Gurvich et al. [2020] Slow $k$-Nim.
Chickin et al. [2021] More about Slow Exact $k$-Nim.

- Two infinite families of games: $\text{SN}(n, 1)$ and $\text{SN}(n, n)$ which are deterministic.
- P-positions are
  - $\text{SN}(n, 1)$: $\text{sum}(p)$ is even.
  - $\text{SN}(n, n)$: $\text{min}(p)$ is even.

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Our Results

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P-positions of a non-trivial infinite family, $\text{SN}(n, n - 1)$, where play is on all but one stack.
Results for P-Positions of Slow Exact $k$-NIM, $k = n - 1$

Characterization of P-positions:

- $s = \text{sum}(p) \mod d$ where $d = k$ when $n$ is even and $d = 2k$ when $n$ is odd
- $o = \# \text{ of stacks with odd stack heights}$

$n = 10, k = 9, s = \text{sum}(p) \mod 9$

$n = 9, k = 8, s = \text{sum}(p) \mod 16$

These results are for REDUCED positions.
Reduced Positions – Motivation – SN(3,2)

\[ p = (1,1,2) \] is the \textbf{reduced} position for \( p = (1,1,2+m) \) with \( m \geq 0 \)

Game trees are isomorphic – same outcome
Reduced Positions - Definition

**Definition:** A position is reduced if, for each stack, there exists a sequence of legal moves that deplete the stack.

Difficult to check, not useful for proofs

**Theorem**
A position is reduced if and only if the **NIRB (No stack Is Really Big) condition**

\[
\max(p) \leq \frac{\text{sum}(p)}{k}
\]

is satisfied.
Outline of Proof of our Results

- Reduce initial position using the NIRB condition repeatedly
- For each position, a move is either to a reduced position or a position that needs to be reduced.
- Characterize when reduction is needed and what reduced position looks like

- Then show that
  - from each P-position, all moves lead to N-positions
  - from each N-position, there is at least one move to a P-position
Reduction Criterion for even \( n \)

\[ \mathbf{p} = (p_1, p_2, \ldots, p_n) \text{ with } p_1 \leq p_2 \leq \ldots \leq p_n \]

**Lemma:** When reduction is needed from a position characterized by \((s, o)\) and \(\mathbf{p}\) has \(\alpha \geq 1\) maximal stacks, then the reduced position is given by

\[
\begin{align*}
    r(\mathbf{p}') &= \begin{cases} 
    (p_1 - 1, p_2 - 1, \ldots, p_n - 1) & \text{if } s > 0 \\
    (p_1 - 1, p_2 - 1, \ldots, p_{n-\alpha} - 1, p_n - 2, \ldots, p_n - 2) & \text{if } s = 0
    \end{cases} 
\end{align*}
\]

- If \( s > 0 \), then \((s', o') = (s - 1, n - o)\); 
- If \( s = 0 \), then \( \alpha \leq n - 2 \) and 
  \[
  s' = n - 2 - \alpha \text{ and } o' = \begin{cases} 
    n - o - \alpha & \text{if } p_n \text{ is even} \\
    n - o + \alpha & \text{if } p_n \text{ is odd}
    \end{cases}
  \]
Illustration for $n$ even – P-Position leads to N-Position

3 cases:
- Move to position that is reduced
- Move to a position that needs reduction; $s > 0$ and $s = 0$

Case 1: No reduction $\rightarrow$ remove exactly $k$ tokens:
- $s' = s \rightarrow$ same row;
- $o' \in \{n - o - 1, n - o + 1\}$ depending on parity of un-played stack
  $\rightarrow$ reflect cell across column $l$ and then go either left or right

$s = \text{sum}(p) \mod k$

$o = \# \text{ of odd stacks}$
Illustration for \( n \) even – P-Position leads to N-Position

**Case 3:** Move is to a position that needs reduction when \( s = 0 \)

- Take a token from all stacks, and an additional token from the maximal stacks

- \( s' = n - 2 - \alpha \) and \( o' = \begin{cases} n - o - \alpha & \text{if } p_n \text{ is even} \\ n - o + \alpha & \text{if } p_n \text{ is odd} \end{cases} \) with \( 1 \leq \alpha \leq n - 2 \)
Illustration for $n$ even: N-position to P-position

- No reduction – same row, reflection across midline, then to left or right depending on parity of max
- Yellow and orange squares are the only ones with potential trouble

For orange squares need to check on reduction move, and it turns out this one is available.

Yellow squares can be shown to have non-reduction move available
Ongoing and Future Work

Generalization to Slow SetNIM SN\((n, A)\), where the set \(A\) indicates the possible numbers of stacks one can play on.

- Develop a NIRB condition for the game with set \(A\)
  \[
  \max(p) \leq \frac{\text{sum}(p)}{k}, \quad \text{where } k = \min(A)
  \]
- Develop characterization when reduction is needed
  - depends on the individual game
- Determine what reduced position looks like
  - depends on the individual game
- Analyze some games
  - we have the result for \(A = \{n - 1, n\}\)
Results for P-Positions of Slow SetNIM $\text{SN}(n, \{n - 1, n\})$

$n = 10, k = 9, s = \text{sum}(p) \mod 9$

$s \backslash o \quad 0 \quad 1 \quad 2 \quad 3 \quad 4 \quad 5 \quad 6 \quad 7 \quad 8 \quad 9 \quad 10$

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$s = \text{sum}(p) \mod 16$

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\(o = \# \text{ of stacks with odd stack heights}\)

- Looks very much like $\text{SN}(n, n - 1)$
- Only difference: P-positions with \(o = n\) are removed
THANK YOU!

Any questions?

You can reach me at sheubac@calstatela.edu
References


Image citation

• Royal Game of Ur, southern Iraq, about 2600-2400 BCE, By BabelStone (Own work), CC0, https://commons.wikimedia.org/w/index.php?curid=10861909
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