Patterns Arising From Tiling Rectangles With 1-by-1 and 2-by-2 Squares

Silvia Heubach
Department of Mathematics and Computer Science
California State University, Los Angeles
5151 State University Drive
Los Angeles, CA 90032-8204, USA
sheubac@calstatela.edu

Phyllis Chinn
Department of Mathematics
Humboldt State University
Arcata, CA 95521
phyllis@math.humboldt.edu

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Abstract

In this paper we look at the number of tilings of an m-by-n area with square tiles of size 1-by-1 (white) and 2-by-2 (red). Let $T^k_{m,n}$ denote the number of these tilings that contain exactly k red squares. We derive recursive and explicit formulas for $T^k_{m,n}$ when $m = 2, 3$. For $m = 4, 5$ we give recursive formulas only. In addition, we present combinatorial proofs for various patterns arising for special values of n and k. Finally, we derive the generating functions for $T^k_{m,n}$ and use them to determine the asymptotic behavior of $T_{m,n}$, the total number of tilings of an m-by-n area.

Keywords: Tiling, Fibonacci numbers, square tiles, generating functions.

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1 Introduction

Tiling questions have often been studied for their connections to well-known number sequences such as the Fibonacci and Lucas sequences [2, 4, 5, 6]; the link between the number of tilings and the respective number sequence then allows for a derivation of well-known identities through combinatorial counting arguments [6, 9]. One basic tiling question is to determine the number of tilings of a 1-by-n area with 1-by-1 and 1-by-2 tiles. These two types of tiles are sometimes referred to in the literature as either squares and dominoes [1, 2, 3, 4, 5], as white and red Cuisenaire© rods ("c-rods") [6, 8, 9, 10, 12, 14] or as monomers and dimers [11, 16, 17]. Various generalizations of this basic tiling question have been investigated: tilings of bracelets [2, 4, 5, 7], random and stacked tilings [1, 3], tilings with other types of c-rods for 1-by-n and for larger rectangular areas [10, 12, 13, 14], as well as extensions to 2-by-2 and larger square tiles [7, 15].

In the literature on tilings of 1-by-n areas with 1-by-1 and 1-by-2 tiles, two approaches were taken to count the total number of tilings. The first of these is a recursive approach, which was extended to count the number of tilings of an m-by-n area with 1-by-1 (white) and 2-by-2 (red) tiles [15]. The second approach used in the c-rod problem was to focus on tilings containing a fixed number of 1-by-2 rods, which yielded many interesting patterns. Motivated by these patterns, we will concentrate on $T^k_{m,n}$, the number of tilings of an m-by-n area with a fixed number of red squares.

Section 2 contains notation and general results for all values of m and n. In Sections 3 and 5 we derive recursive and explicit formulas for $T^k_{m,n}$, while Sections 4 and 6 contain results on patterns for specific values of n and k and their combinatorial proofs. In Section 7 we discuss extensions of the results to $m > 5$. Finally, in Section 8, we derive the generating functions for $T^k_{m,n}$ and $T_{m,n}$, the total number of tilings, which are then used to establish asymptotic results for $T_{5,n}$ and $T_{4,n}$.

2 Notation and General Results

We use the following notation, where an m-by-n rectangle has m rows and n columns:

$T_{m,n}$	$=$	the total number of tilings of an m-by-n rectangle with 1-by-1 (white) and 2-by-2 (red) squares
$T_{m,n}^k$	$=$	the number of tilings of an m-by-n rectangle with white and red squares which contain **exactly** k red squares
F_n	$=$	the n^{th} Fibonacci number ($F_1 = F_2 = 1$, $F_m = F_{m-1} + F_{m-2}$).

In addition, we define $T_{m,0}^0 = 1$ and $T_{m,0}^k = 0$ for $k \geq 1$ and all m. Before we derive results for specific values of m, here are some general results:

Theorem 1

1. $T_{m,n} = \sum_{k \geq 0} T_{m,n}^k$, *where* $T_{m,n}^k = 0$ *for* $k > \lfloor (m \cdot n)/4 \rfloor$, $n \geq 1$.
2. $T_{m,n}^1 = (m-1)(n-1)$ *for* $m, n \geq 1$.
3. $T_{m,n}^0 = 1$ *for all values of* m *and* n.

Proof:

1. The first statement just expresses the fact that the total number of tilings can be obtained by first counting the number of tilings with each possible number of red tiles, then summing with respect to the number of red tiles. At most $(m \cdot n)/4$ red squares can be placed in a tiling of size m-by-n.
2. The lower left corner of a red square can be placed in any of $m - 1$ rows and $n - 1$ columns.

3. In this case, the tiling consists of only white tiles. □

3 Results for $T^k_{2,n}$ and $T^k_{3,n}$

For $m = 2, 3$, both recursive and exact results can be obtained for $T^k_{m,n}$. The two cases are very similar in structure, as evidenced in the theorem below.

Theorem 2 *The recursive and explicit formulas for the number of tilings of a 2-by-n and 3-by-n area, respectively, containing exactly k red squares are given by*

1. $T^k_{2,n} = T^k_{2,n-1} + T^{k-1}_{2,n-2}$ *for* $n \geq 2, k \geq 1$ *and* $T^k_{2,n} = \binom{n-k}{k}$ *for* $n \geq 2k$.
2. $T^k_{3,n} = T^k_{3,n-1} + 2 \cdot T^{k-1}_{3,n-2}$ *for* $n \geq 2, k \geq 1$ *and* $T^k_{3,n} = \binom{n-k}{k} \cdot 2^k$, *for* $n \geq 2k$.

Proof:

1. Since each tiling of a 2-by-n rectangle with white and red squares is in a one-to-one correspondence to a tiling of a 1-by-n rectangle using white (1-by-1) and red (1-by-2) c-rods as indicated in Figure 1, the results for $T^k_{2,n}$ follow from [6].

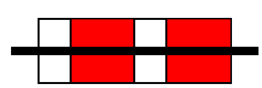

Figure 1: One-to-one Correspondence

2. To derive the recursive formula for $T^k_{3,n}$, we look at the first column. If no red square occurs in the first column, then k red squares have to occur in a tiling of size 3-by-(n-1), contributing $T^k_{3,n-1}$ tilings. If, on the other hand, a red square starts in the first column, then there need to be $k-1$ red tiles in the remaining

$n-2$ columns. Furthermore, the square in the first column can occur in one of two places, either at the bottom or at the top. Overall, there are $2 \cdot T_{3,n-2}^{k-1}$ such tilings.

The explicit formula for $T_{3,n}^k$ is derived as follows: Consider the tiling as a line-up consisting of white stacks (three vertically aligned white squares) and mixed stacks (one red square combined with two white squares, covering two columns). Thus, a tiling containing k red squares is a line-up of $n-k$ objects, and the k mixed stacks can be placed in $\binom{n-k}{k}$ ways. For each of these k mixed stacks there are two possible arrangements (white squares at top or bottom), resulting in the factor 2^k. Finally, in order to place k red squares, we need $n \geq 2k$. □

In the next section we will look at tables of values for $T_{2,n}^k$ and $T_{3,n}^k$, and give combinatorial proofs for some patterns within these tables.

4 Patterns for $T_{2,n}^k$ and $T_{3,n}^k$

We will first look at patterns for $T_{2,n}^k$. Several patterns are evident in Table 1 in addition to the universal pattern in the column for $k=1$, which was shown in Theorem 1, part (2).

Theorem 3

1. *The l^{th} diagonal of slope -1 contains the values of the l^{th} row of Pascal's triangle (if we count the single 1 as the 0^{th} row).*

2. *The values in the l^{th} diagonal of slope -2 equal the values in the column for $k=l$, $k \geq 0$, i.e. $T_{2,l+2k}^k = T_{2,2l+k}^l$.*

k / n	0	1	2	3	4	5	6
0	1						
1	1						
2	1	1					
3	1	2					
4	1	3	1				
5	1	4	3				
6	1	5	6	1			
7	1	6	10	4			
8	1	7	15	10	1		
9	1	8	21	20	5		
10	1	9	28	35	15	1	
11	1	10	36	56	35	6	
12	1	11	45	84	70	21	1

Table 1: Values for $T_{2,n}^k$

Proof:

All of these patterns can be shown using the exlicit formula derived in Theorem 2. However, we will give combinatorial arguments for each.

1. Entries in the l^{th} diagonal of slope -1 are of the form $T_{2,l+k}^k$ for $l \geq 0$, i.e., we need to line up $n - k = (l + k - k) = l$ objects (white stacks and red squares) in sequence (analogous to the proof of Theorem 2 for $m = 3$). There are $\binom{l}{k}$ ways to select the positions of the k red squares.

2. The entries in the l^{th} diagonal of slope -2 are of the form $T_{2,l+2k}^k$. Once more we use the argument that for a tiling with k red squares, a total of $n - k$ white stacks and red squares need to be placed. Thus, for $n = l + 2k$, there are $l + k$ objects, more specifically k red squares and l stacks. We start with a tiling of length $l + 2k$ with k red squares, and replace each red square by a stack and vice versa to obtain a tiling with l red tiles of length $l + 2k - k + l = 2l + k$. This

process is shown in Figure 2 for the case $l = 1, k = 2$, where $T^2_{2,5} = T^1_{2,4} = 3$.

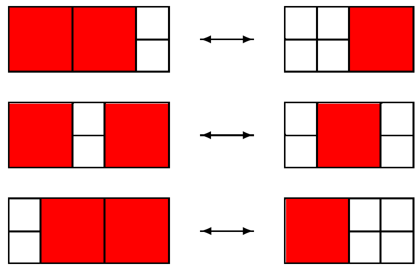

Figure 2: Interchanging Red Squares and White Stacks

Now we turn our attention to patterns for $T^k_{3,n}$:

k / n	0	1	2	3	4	5	6
0	1						
1	1						
2	1	2					
3	1	4					
4	1	6	4				
5	1	8	12				
6	1	10	24	8			
7	1	12	40	32			
8	1	14	60	80	16		
9	1	16	84	160	80		
10	1	18	112	280	240	32	
11	1	20	144	448	560	192	
12	1	22	180	672	1120	672	64

Table 2: Values for $T^k_{3,n}$

We can verify the following new patterns apparent in Table 2:

Theorem 4

1. $T^2_{3,n} = 2(n-2)(n-3),\ n \geq 4.$

2. $T^k_{3,2k} = 2^k,\ k \geq 1.$

3. $T^k_{2,l+2k} = T^l_{2,2l+k} \cdot 2^{k-l}.$

Proof:

The first two statements follow immediately from the explicit formula given in Theorem 2. The third statement relates elements in the l^{th} diagonal to those in the column for $k = l$, as in the case $m = 2$. However, due to the fact that there are two types of mixed stacks containing one red square, there now has to be a factor of a power of 2 that adjusts for this fact. □

5 Results for $T^k_{4,n}$ and $T^k_{5,n}$

Unlike the cases $m = 2, 3$, the cases $m = 4, 5$ only yield a recursive formula.

Theorem 5 *The number of tilings of a 4-by-n and 5-by-n area, respectively, with exactly k red squares is given by*

$$T^k_{m,n} = T^k_{m,n-1} + \sum_{l=1}^{2} \binom{m-l}{l} \cdot T^{k-l}_{m,n-2} + \sum_{r=3}^{min\{k+1,n\}} B_{m,r} \cdot T^{k-r+1}_{m,n-r} \qquad \textit{for } n, k \geq 2$$

where $B_{4,r} = 2$ *and* $B_{5,r} = 2 \cdot F_{r+1}$ *for* $r \geq 3$.

Proof:

There are several ways in which the tiling can start out in the first column:

1. No red tile in the first column implies that k red tiles need to occur in the remaining rectangle of size m-by-(n-1), resulting in $T^k_{m,n-1}$ tilings.

2. If there are l red tiles in the first column ($1 \leq l \leq 2$) and no red tile whose lower left corner is in the second column, then $k-l$ red tiles need to occur in a rectangle of size m-by-$(n-2)$. In addition, the red tiles in the first column can be placed in $\binom{m-l}{l}$ ways, so there are $\binom{m-l}{l}T^{k-l}_{m,n-l}$ tilings.

3. The last possibility for a tiling is to have one red tile in the first column and one red tile whose lower left corner is in the second column. This starts an "interlocking" pattern which will cover r columns, where $3 \leq r \leq n$.

   For $m = 4$, there are exactly two interlocking patterns of any given length r as indicated in Figure 3.

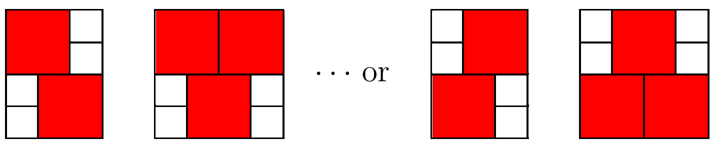

Figure 3: Interlocking Patterns

   For $m = 5$, there is a two-to-one correspondence to tilings with c-rods, i.e., the number of ways to create an interlocking pattern spanning r columns is twice the number of ways to tile a 1-by-r rectangle with white (1-by-1) and red (1-by-2) c-rods, which can be done in F_{r+1} ways [6]. Here is how it works: For a given interlocking pattern, look at the middle row, which reduces to a tiling of a 1-by-r area with red and white c-rods, as shown in Figure 4.

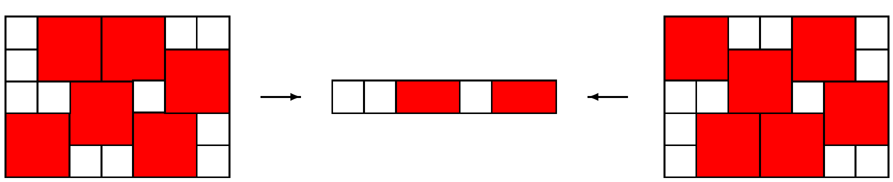

Figure 4: Reduction Process

   On the other hand, given a tiling with c-rods, exactly two tilings with an interlocking pattern can be created. If the c-rod tiling starts with a white tile, then a red square has to be put either above or below (since in each column, a red tile starts). If the c-rod tiling starts with a red c-rod, then this c-rod has be extended either upwards or downwards into a square. Once the initial decision has been made, there is only one way to finish the interlocking pattern, as a red tile starting in column j occupies or extends to either the upper or lower half. The red tile starting in column $j + 1$ needs to be placed into or extended to the opposite half. Any remaining empty spaces are filled with white tiles. The steps in this "creation" process for one of the two possibilities in Figure 4 are shown in Figure 5.

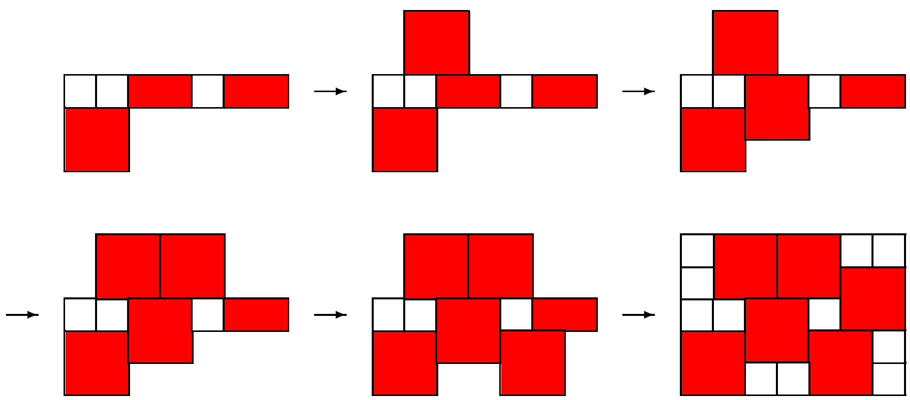

Figure 5: Creation Process

Note that for both $m = 4$ and $m = 5$, an interlocking pattern spanning r columns contains exactly $r - 1$ red squares. The remaining $k - (r - 1)$ red squares have to occur in a rectangle of size m-by-$(n-r)$. The largest interlocking pattern consists of k red squares ($r = k + 1$); altogether, there are a total of $\sum_{r=3}^{\min\{k+1,n\}} T_{m,n-r}^{k-(r-1)}$ tilings. □

6 Patterns for $T^k_{4,n}$ and $T^k_{5,n}$

We will follow the format of Section 4 and present tables of values and patterns within those tables for $T^k_{4,n}$ and $T^k_{5,n}$. We start with a table of values for $T^k_{4,n}$:

k / n	0	1	2	3	4	5	6	7	8	9	10
0	1										
1	1										
2	1	3	1								
3	1	6	4								
4	1	9	16	8	1						
5	1	12	37	34	9						
6	1	15	67	105	65	15	1				
7	1	18	106	248	250	108	16				
8	1	21	154	490	726	522	176	24	1		
9	1	24	211	858	1736	1824	994	260	25		
10	1	27	277	1379	3604	5148	4090	1770	385	35	1

Table 3: Values for $T^k_{4,n}$

In addition to the pattern in the column for $k = 1$ indicated in Theorem 1, the following patterns can be identified in Table 3:

Theorem 6

1. $T^{2k}_{4,2k} = 1,\ k \geq 0.$

2. $T^{2k-1}_{4,2k} = (k+1)^2 - 1,\ k \geq 0.$

3. $T^{2k}_{4,2k+1} = (k+1)^2,\ k \geq 1.$

Proof:

1. As $2k$ red tiles is the maximum possible number of tiles for a 4-by-$2k$ rectangle, there is exactly one tiling (consisting of only red squares).

2. Again we are looking at a 4-by-$2k$ rectangle, but now we place one fewer than the maximum number of red squares possible. This means that four white tiles occur within the tiling. Due to the geometry of the red tiles, the white tiles occur as two stacks, which can be placed either horizontally or vertically. If both stacks are placed vertically, then they both have to be in either the upper or the lower half of the tiling (since the number of columns is even). The half in which the white stacks are located contains $k-1$ red tiles, thus there are $\binom{(k-1)+2}{2}$ possible ways to place those white stacks. By symmetry there are $2\frac{(k+1)k}{2} = k^2 + k$ such tilings. If, on the other hand, the stacks are placed horizontally, they both have to be in the same two columns. There are three possible locations for the red tile covering the remaining two rows. However, two of the three possibilities have already been counted above, leaving just the k tilings in which the white stacks are separated by the red tile. Overall, there are $k^2 + 2k = (k+1)^2 - 1$ tilings.

3. As in the second case, we are placing $2k$ red tiles, but now there is an additional column, which means that the tiling contains four white tiles. Since the number of columns is odd, these white tiles have to occur as two vertical stacks, one in the upper two rows and the other stack in the lower two rows. Each stack can be placed in $k+1$ ways among the k red squares in its half which implies there are $(k+1)^2$ tilings. $\square$

Next we consider the case m = 5. Again we start by displaying a table of values for $T^k_{5,n}$:

n \ k	0	1	2	3	4	5	6	7	8
0	1								
1	1								
2	1	4	3						
3	1	8	12						
4	1	12	37	34	9				
5	1	16	78	140	79				
6	1	20	135	382	454	194	27		
7	1	24	208	824	1566	1344	408		
8	1	28	297	1530	4103	5670	3698	926	81

Table 4: Values for $T^k_{5,n}$

The next theorem states the only obvious pattern in Table 4 besides the universal pattern for the column $k = 1$.

Theorem 7 $\quad T^{2k}_{5,2k} = 3^k, \ k \geq 1.$

Proof: Since we are placing $2k$ tiles into $2k$ columns, they cannot form an interlocking pattern (those have $r-1$ tiles in r columns). This means that each of the k pairs of two consecutive columns must contain exactly two red tiles, which can be placed in 3 ways within those two columns yielding 3^k possible tilings. □

7 Generalizations

The recursive formulas of Theorems 2 and 5 can be combined in the following general formula:

Corollary 8 *The number of tilings of a m-by-n area with exactly k red squares is given (recursively) by*

$$T^k_{m,n} = T^k_{m,n-1} + \sum_{l=1}^{\lfloor m/2 \rfloor} \binom{m-l}{l} \cdot T^{k-l}_{m,n-2} + \sum_{r=3}^{min\{k+1,n\}} B_{m,r} \cdot T^{k-r+1}_{m,n-r}$$

for $n \geq 2, k \geq \lfloor m/2 \rfloor, m = 2, ..., 5,$ *where* $B_{2,r} = B_{3,r} = 0, B_{4,r} = 2$ *and* $B_{5,r} = F_{r+1}$ *for* $r \geq 3$.

Proof:

The formula above is identical to Theorem 5 for $m = 4, 5$ and reduces to the recursive results of Theorem 2 by noting that there are no interlocking patterns for $m = 2, 3$, i.e. $B_{2,r} = B_{3,r} = 0$. □

The question now becomes whether a generalization of this formula is valid for values of $m > 5$ as well. At first glance it seems as if one could extend the reasoning very easily. Looking at the tilings that do not start with an interlocking pattern, the argument is already valid for all m. However, counting the tilings that start with an interlocking pattern is much more difficult, mainly because for $m > 5$, the number of tiles that occur in an interlocking pattern of length r are no longer fixed. Figure 6 shows examples of tilings with interlocking patterns for $r = 3$ and $m = 6$, which contain a varying number of red tiles.

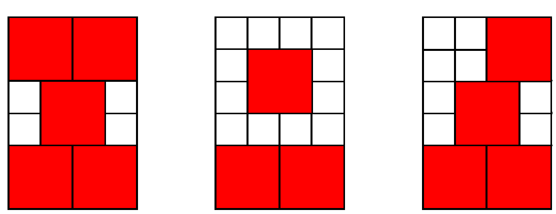

Figure 6: Interlocking Patterns for $m = 6$ and $r = 3$

This inability to know the exact number of red tiles that have to occur in the remainder of the tiling based on the width of the interlocking pattern prohibits a straight-forward extension of the formula given in Corollary 8.

8 Generating Functions and Asymptotic Results

We will now derive the generating functions for $\left\{T^k_{m,n}\right\}_{n,k\geq 0}$ and $\{T_{m,n}\}_{n\geq 0}$. Even though Corollary 8 provides a unified formula for $m = 2, ..., 5$, we will group the cases $m = 2, 3$ and $m = 4, 5$. One reason for this grouping is that these two sets of cases have different ranges of validity. Secondly, explicit formulas for $T_{m,n}$ for $m = 2, 3$ were derived using a different approach in [15]. For completeness, we state the result from this paper (without proof) in the next theorem.

Theorem 9

1. *$T_{2,n} = F_{n+1}$ for $n \geq 0$.*

2. *$T_{3,n} = (2^{n+1} - (-1)^{n+1})/3$ for $n \geq 1$.*

Before we prove the results for the generating functions, we will state three results that will be useful for the proofs in the remainder of this section. We use the following notation:

$$G_m(x) = \sum_{n=0}^{\infty} T_{m,n} x^n \quad \text{and} \quad g_m(x,t) = \sum_{n=0}^{\infty}\sum_{k=0}^{\infty} T^k_{m,n} x^n t^k$$

Lemma 10

1. *$\sum\limits_{n\geq 0} T^0_{m,n} x^n = \frac{1}{x-1}$.*

2. $\sum_{n\geq 0} T^1_{m,n} x^n t = \frac{(m-1)x^2 t}{(1-x)^2}$.

3. $G_m(x) = g_m(x, 1)$.

Proof: We first state the relevant initial conditions (which hold for all m, n unless otherwise stated) needed in this proof and the ones that follow:

$T^0_{m,n} = 1, T^1_{m,0} = T^1_{m,1} = 0, T^1_{m,n} = (m-1)(n-1)$ for $n \geq 2$, and $T^k_{m,0} = 0$ for $k \geq 1$.

With these initial conditions

$$\sum_{n\geq 0} T^0_{m,n} x^n = \sum_{n\geq 0} x^n = \frac{1}{x-1}$$

and

$$\begin{aligned} \sum_{n\geq 0} T^1_{m,n} x^n t &= xt(m-1) \sum_{n\geq 2} (n-1) x^{n-1} = xt(m-1) \sum_{n\geq 1} n x^n \\ &= xt(m-1) x D(\frac{1}{1-x}) = \frac{(m-1)x^2 t}{(1-x)^2} \end{aligned}$$

where D is the differential operator, and the second-to-last equality follows from rules for generating functions (see for example [19]). Statement 3 follows immediately from Theorem 1, part (1) and the definitions of $G_m(x)$ and $g_m(x, t)$. □

Theorem 11 *For $m = 2$ and $m = 3$, the generating functions for $T^k_{m,n}$ and $T_{m,n}$, respectively, are given by*

$$g_m(x, t) = \frac{1}{1 - x - (m-1)x^2 t} \quad \textit{and} \quad G_m(x) = \frac{1}{1 - x - (m-1)x^2}.$$

Proof: The result for $g_m(x, t)$ can be derived similarly to the more complex cases shown in the proof of Theorem 12. The recurrence equation reduces to

$$T^k_{m,n} = T^k_{m,n-1} + (m-1) T^{k-1}_{m,n-2} \quad \text{for} \quad n \geq 2, k \geq 1.$$

Multiply each term in this equation by $x^n t^k$ and sum over $n \geq 2, k \geq 1$. Using appropriate initial conditions (see proof of Lemma 10) and algebraic manipulations similar to those in the proof of Theorem 12, the results for $g_{m,n}(x)$ and $G_m(x)$ follow from Lemma 10. □

We will now look in greater depth at the cases $m = 4, 5$.

Theorem 12 *The generating functions for $T^k_{m,n}$ and $T_{m,n}$, respectively, for $m = 4, 5$ are given by*

1. $g_4(x,t) = \frac{1-xt}{(1-xt)(1-x-3x^2t-x^2t^2)-2x^3t^2}$ *and* $G_4(x) = \frac{1-x}{1-2x-3x^2+2x^3}$

2. $g_5(x,t) = \frac{1-xt-x^2t^2}{1-x-xt-3x^2t-4x^2t^2-x^3t^2+3x^3t^3+3x^4t^4}$ *and* $G_5(x) = \frac{1-x-x^2}{1-2x-7x^2+2x^3+3x^4}$.

Proof: For $m = 4, 5$, the recursion given in Corollary 8 can be written as

$$T^k_{m,n} = T^k_{m,n-1} + (m-1) \cdot T^{k-1}_{m,n-2} + \binom{m-2}{2} \cdot T^{k-2}_{m,n-2} + \sum_{r=3}^{\min\{k+1,n\}} B_{m,r} \cdot T^{k-r+1}_{m,n-r} \quad (1)$$

for $n, k \geq 2$, where $B_{4,r} = 2$ and $B_{5,r} = F_{r+1}$. Multiply each term of the recursion by $x^n t^k$ and sum over $n, k \geq 2$. After factoring out appropriate powers of x and t, so that powers and subscripts match, and renaming the summation indicies, Equation (1) becomes the following:

$$\begin{aligned} \sum_{n\geq 2}\sum_{k\geq 2} T^k_{m,n} x^n t^k \quad = \quad & x \sum_{\tilde{n}\geq 1}\sum_{k\geq 2} T^k_{m,\tilde{n}} x^{\tilde{n}} t^k + (m-1)x^2 t \sum_{\tilde{n}\geq 0}\sum_{\tilde{k}\geq 1} T^{\tilde{k}}_{m,\tilde{n}} x^{\tilde{n}} t^{\tilde{k}} \\ & + \binom{m-2}{2} x^2 t^2 \sum_{\tilde{n}\geq 0}\sum_{\tilde{k}\geq 0} T^{\tilde{k}}_{m,\tilde{n}} x^{\tilde{n}} t^{\tilde{k}} \\ & + \sum_{n\geq 2}\sum_{k\geq 2} \sum_{r=3}^{\min\{k+1,n\}} B_{m,r} \cdot T^{k-r+1}_{m,n-r} x^n t^k. \end{aligned} \quad (2)$$

Using the definition of $g_m(x,t)$ and the initial conditions stated in the proof of Lemma 10, Equation (2) reduces to

$$\begin{aligned} g_m(x,t) \quad - & \quad \sum_{n\geq 0} T^0_{m,n} x^n t^0 - \sum_{n\geq 0} T^1_{m,n} x^n t^1 = \\ & x\left(g_m(x,t) - \sum_{n\geq 0} T^0_{m,n} x^n t^0 - \sum_{n\geq 0} T^1_{m,n} x^n t^1\right) \\ & +(m-1)x^2 t\left(g_m(x,t) - \sum_{n\geq 0} T^0_{m,n} x^n t^0\right) \\ & +\binom{m-2}{2} x^2 t^2 g_m(x,t) + \sum_{n\geq 2}\sum_{k\geq 2}\sum_{r=3}^{\min\{k+1,n\}} B_{m,r}\cdot T^{k-r+1}_{m,n-r} x^n t^k. \qquad (3) \end{aligned}$$

All the sums except the triple sum can be dealt with using Lemma 10. For the triple sum, we need to change the order of summation. Since $3 \leq r \leq k+1$ and $3 \leq r \leq n$, we get that $n \geq r$ and $k \geq r-1$. The limits for r become $3 \leq r < \infty$. Thus, after factoring out appropriate powers of x and t, and renaming the summation indices, we get

$$\begin{aligned} \sum_{n\geq 2}\sum_{k\geq 2}\sum_{r=3}^{\min\{k+1,n\}} B_{m,r}\cdot T^{k-r+1}_{m,n-r} x^n t^k &= \sum_{r\geq 3} B_{m,r}\cdot x^r t^{r-1} \sum_{n\geq r}\sum_{k\geq r-1} T^{k-r+1}_{m,n-r} x^{n-r} t^{k-r+1} \\ &= \sum_{r\geq 3} B_{m,r}\cdot x^r t^{r-1} \sum_{\tilde{n}\geq 0}\sum_{\tilde{k}\geq 0} T^{\tilde{k}}_{m,\tilde{n}} x^{\tilde{n}} t^{\tilde{k}} \\ &= \sum_{r\geq 3} B_{m,r}\cdot x^r t^{r-1} g_m(x,t). \qquad (4) \end{aligned}$$

We will now look at the last sum, where we finally have to make the distinction between $m=4$ and $m=5$. For $m=4$, $B_{m,r}=2$, and we get

$$\sum_{r\geq 3} B_{m,r}\cdot x^r t^{r-1} = 2x^3t^2 \sum_{\tilde{r}\geq 0} (xt)^{\tilde{r}} = \frac{2x^3t^2}{1-xt}. \qquad (5)$$

For $m=5$, $B_{m,r} = 2F_{r+1} = 2\frac{1}{\sqrt{5}}((\phi_+)^{r+1} - (\phi_-)^{r+1})$, where $\phi_\pm = \frac{1\pm\sqrt{5}}{2}$. Thus,

$$\sum_{r\geq 3} B_{m,r} x^r t^{r-1} \quad = \quad \frac{2x^3t^2}{\sqrt{5}} \sum_{r\geq 3} ((\phi_+)^{r+1} - (\phi_-)^{r+1})(xt)^{r-3}$$

$$
\begin{aligned}
&= \frac{2x^3t^2}{\sqrt{5}} \left\{ (\phi_+)^4 \sum_{\tilde{r}\geq 0} ((\phi_+)xt)^{\tilde{r}} - (\phi_-)^4 \sum_{\tilde{r}\geq 0} ((\phi_-)xt)^{\tilde{r}} \right\} \\
&= \frac{2x^3t^2}{\sqrt{5}} \left\{ (\phi_+)^4 \frac{1}{1-(\phi_+)xt} - (\phi_-)^4 \frac{1}{1-(\phi_-)xt} \right\} \\
&= \frac{(2x^3t^2)(3+2xt)}{(1-xt-x^2t^2)}. \qquad (6)
\end{aligned}
$$

Substituting Equations (5) and (6), respectively, into Equation (4) and combining the result with Equation (3) and with Lemma 10, part (1) and (2) leads to equations in $g_4(x,t)$ and $g_5(x,t)$. Solving these equations for $g_4(x,t)$ and $g_5(x,t)$, respectively, leads to the desired results. The formulas for $G_4(x)$ and $G_5(x)$ follow from Lemma 10, part (3). □

Since only recursive results were derived for $T_{4,n}$ and $T_{5,n}$, we will use the respective generating function to determine the asymptotic behavior of $T_{4,n}$ and $T_{5,n}$. We will consider the generating functions as functions on the complex plane, and will use the variable z to emphasize this fact. This change of viewpoint will allow us to use facts regarding complex functions. In particular, Theorem 5.2.1 [19] states that the coefficients of the generating function G_m can be approximated by the coefficients of the sum of the prinicipal parts of the Laurent series of G_m about its poles. The first step is to identify the poles of G_m.

Lemma 13 *The three simple poles of* $G_4(z) = \frac{1}{1-2z-3z^2+2z^3}$ *occur at*

$$
\begin{aligned}
z_1 &= \frac{1}{2} - \sqrt{\frac{7}{3}} Cos\left(\frac{1}{3}\left\{2\pi - ArcCos\left(-\frac{3\sqrt{\frac{3}{7}}}{7}\right)\right\}\right) = 0.35541572677584501545 8.. \\
z_2 &= \frac{1}{2} - \sqrt{\frac{7}{3}} Cos\left(\frac{1}{3} ArcCos\left(-\frac{3\sqrt{\frac{3}{7}}}{7}\right)\right) = -0.744644285905039381396.. \\
z_3 &= \frac{1}{2} + \sqrt{\frac{7}{3}} Cos\left(\frac{1}{3} ArcCos\left(\frac{3\sqrt{\frac{3}{7}}}{7}\right)\right) = 1.88922855912919436593 7....
\end{aligned}
$$

Proof: This follows from classical methods of solving cubic equations. For example, see [18], pp.184-185. □

We can now state the asymptotic result for $T_{4,n}$:

Theorem 14 *With the notation of Lemma 13, for every* $\epsilon > 0$*,*

$$T_{4,n} = \frac{-b_1}{z_1^{n+1}} + \frac{-b_2}{z_2^{n+1}} + O\left(\left(\frac{1}{|z_3|} + \epsilon\right)^n\right) \approx \frac{0.191012}{z_1^{n+1}} - \frac{0.301069}{z_2^{n+1}}$$

where $b_i = \lim_{z \to z_i} G_4(z)(z - z_i)$*.*

Proof: Since each pole of $G_4(z)$ is simple, the principal part of the Laurent series expansion about the pole z_i, denoted by $PP(G_4; z_i)$, is of the form $\frac{b_i}{(z-z_i)}$, where $b_i = \lim_{z \to z_i} G_4(z)(z - z_i)$. Using Theorem 5.2.1 [19] twice to include all poles with modulus less than or equal to one for a good approximation, we get that

$$T_{4,n} = [z^n]\, G_4(z) = [z^n]\, \{PP(G_4; z_1) + PP(G_4; z_2)\} + O\left(\left(\frac{1}{|z_3|} + \epsilon\right)^n\right)$$

where $[z^n]\, G_4(z)$ denotes the coefficient of z^n in the power series exapnaion of $G_4(z)$. Since

$$PP(G_4; z_i) = \frac{b_i}{(z - z_i)} = \frac{-b_i}{z_i(1 - \frac{z}{z_i})} = \frac{-b_i}{z_i} \sum_{n=0}^{\infty} \left(\frac{z}{z_i}\right)^n = \sum_{n=0}^{\infty} \left(\frac{-b_i}{z_i^{n+1}}\right) z^n$$

the statement follows. □

If the exact values given in Lemma 13 are used, then the approximation gives very good results even for small n, and the correct integer values for $n \geq 55$. Table 5 shows the actual values versus the approximation (rounded to four decimal places).

n	$T_{4,n}$	$\frac{-b_1}{z_1^{n+1}} + \frac{-b_2}{z_2^{n+1}}$
1	1	0.9692
3	11	10.9914
5	93	92.9976
10	16717	16716.9999
30	16151954937485	16151954937485.0000
55	2753018120127920732449067	2753018120127920732449067

Table 5: Asymptotics for $T_{4,n}$

Finally, we will look at the asymptotics for $T_{5,n}$, using the same aproach as for $m = 4$. Again, we need to determine the poles of the generating function. However, in this case we cannot derive a closed form for the zeros of the denominator of $G_5(z)$, and have to be satisfied with numerical approximations produced by Mathematica or a similar computer algebra system.

Lemma 15 *The four simple poles of* $G_5(z) = \frac{1-z-z^2}{1-2z-7z^2+2z^3+3z^4}$ *occur at*

$$
\begin{aligned}
z_1 &= 0.270980205753790415200817467926505182959 9....\\
z_2 &= -0.536104761934466671556658261302814950300....\\
z_3 &= 1.327243095572559525099270109115332887049....\\
z_4 &= -1.728785206058549935410095982405689786376....
\end{aligned}
$$

We can now get asymptotic results for $T_{5,n}$ by using the two poles with magnitude less than or equal to one.

Theorem 16 *With the notation of Lemma 15, for every* $\epsilon > 0$,

$$T_{5,n} = \frac{-b_1}{z_1^{n+1}} + \frac{-b_2}{z_2^{n+1}} + O\left(\left(\frac{1}{|z_3|} + \epsilon\right)^n\right) \approx \frac{0.128186}{z_1^{n+1}} - \frac{0.232059}{z_2^{n+1}}$$

where $b_i = \lim_{z \to z_i} G_5(z)(z - z_i)$.

Proof: The proof is identical to the one of Theorem 14, with $G_4(z)$ replaced by $G_5(z)$. □

If the values given in Lemma 15 are used with a precision of 40, then the approximation again gives very good results even for small n, and gives the correct integer values for $n \geq 58$. Table 6 shows the actual values versus the approximation (rounded to four decimal places).

n	$T_{4,n}$	$\frac{-b_1}{z_1^{n+1}} + \frac{-b_2}{z_2^{n+1}}$
1	1	0.9383
3	21	20.9640
5	314	313.9793
10	221799	221798.9948
30	48615617121891973	48615617121891973.0000
58	366879878040417912475225104875511	366879878040417912475225104875511

Table 6: Asymptotics for $T_{5,n}$

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