Spring 2019 Topology Comprehensive Exam

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Do any five (5) of the problems that follow. Each is worth 20 points. Please indicate clearly which five you want us to grade. (Otherwise, we will grade your first five answers.)

Unless otherwise stated, we endow $\mathbb{R}$ with the usual (Euclidean) topology. For real numbers a, b with $a < b$, we denote by $[a, b]$ the closed interval from a to b, regarded as a subspace of $\mathbb{R}$.

(1) Let X and Y be topological spaces. Let A and B be subsets of X, each endowed with the subspace topology. For each of the following statements, indicate whether it is always true, and prove that your answer is correct.

    (a) If $f\colon X \to Y$ is continuous, then $f|A$ is continuous. Here $f|A$ denotes the restriction of f to A, that is, the function $f|A\colon A \to Y$ defined by $(f|A)(x) = f(x)$ for all $x \in A$.

    (b) If f is a function from X to Y, and $X = A \cup B$, and $f|A$ and $f|B$ are continuous, then f is continuous.

(2) We say a topological space S has the *fixed-point property* if whenever $f\colon S \to S$ is continuous, there exists $x \in S$ such that $f(x) = x$.

    (a) Prove that if S has the fixed point property, then S is connected.

    (b) Does every connected topological space have the fixed-point property? Prove that your answer is correct.

    (c) Prove that $[0, 1]$ has the fixed-point property. Hint: Use the Intermediate Value Theorem.

(3) Let $(S_1, d_1), (S_2, d_2), \ldots, (S_k, d_k)$ be metric spaces. Prove that the product space $S_1 \times S_2 \times \cdots \times S_k$ is metrizable.

(4) Let X be an infinite set. Suppose we have a Hausdorff topology on X. Prove that some subset of X must be non-compact. Hint: Do a proof by contradiction.

(5) We say that a topological space X is a T_3-*space* if:

    (a) Whenever $x \in X$ and A is a closed subset of X, there exist disjoint open sets U and V such that $x \in U$ and $A \subseteq V$, and

    (b) For all $x \in X$, the set $\{x\}$ is closed.

    Prove that every subspace of a T_3-space is a T_3-space.

(6) Let $X = \{(t,t) \mid -1 \le t \le 1\} \cup \{(t,-t) \mid -1 \le t \le 1\}$. Regard X as a subspace of $\mathbb{R}^2$. (If you draw a picture of X, it should look like the letter X.) Prove that X is not homeomorphic to the closed interval $[0,1]$.

(7) Let $X = [1,3] \times [1,3]$. We equip X with the "lexicographic" order:

$(x_1, y_1) \le (x_2, y_2)$ if and only if either $x_1 < x_2$, or else $x_1 = x_2$ and $y_1 < y_2$.

    We endow X with the corresponding order topology.

    (a) Is $\{(x,y) \in X \mid 1 < x < 3\}$ an open subset of X? Prove that your answer is correct.

    (b) Is $\{(x,y) \in X \mid y = 2.7\}$ a closed subset of X? Prove that your answer is correct.

    (c) Is $\{(x,y) \in X \mid y = x\}$ a connected subset of X? Prove that your answer is correct.