Spring 2013 Topology comprehensive exam

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Please do any FIVE of the seven problems below. They are worth 20 points each. Indicate CLEARLY which five you want us to grade—if you do more than five problems, we will select five to grade, and they may not be the five that you want us to grade.

Let $\mathbb{R}$ denote the set of real numbers. Unless otherwise stated, assume that $\mathbb{R}^n$ is endowed with the usual topology; that subsets of a given topological space are endowed with the subspace topology; and that products are endowed with the product topology. We use the notation "∂A" for the boundary of a set A.

(1) Suppose X, Y are topological spaces, and suppose that X is compact and Y is Hausdorff. Prove that if $f : X \to Y$ is bijective and continuous, then f is a homeomorphism.

(2) Prove that $\mathbb{R}^2$ has a countable basis.

(3) Prove that a topological space S is connected if and only if $\partial H \neq \emptyset$ for every nonempty proper subset H of S.

(4) Let $\langle X, d\rangle$ be a metric space.

(a) Suppose X is compact. Prove that for each $\varepsilon > 0$ there exists a finite subset F of X such that for each $x \in X$, there exists $y \in F$ with $d(x, y) < \varepsilon$.

(b) Consider $X = (0, 1)$ equipped with the metric $d(x, y) = |x - y|$. Show that X is noncompact, yet the property described in (a) still holds.

(5) Consider a nonempty set X equipped with the discrete topology.

(a) Prove that X is normal.

(b) Give necessary and sufficient conditions for X to be compact, and prove that your answer is correct.

(c) Give necessary and sufficient conditions for X to be connected, and prove that your answer is correct.

(6) Give counterexamples to each statement below.

(a) If $f : \mathbb{R} \to \mathbb{R}$ and $\{(x, f(x)) \mid x \in \mathbb{R}\}$ is closed in $\mathbb{R}^2$, then f is continuous.

(b) If $f : \mathbb{R} \to \mathbb{R}$ is continuous and V is an open subset of R, then $f(V)$ is open as well.

(c) If $A \subseteq B \subseteq \mathbb{R}$, then $\partial(A) \subseteq \partial(B)$.

(7) Let A be a nonempty path-connected topological space. Let $x \in A$. Prove that $(A \times A)\setminus\{(x, x)\}$ is path-connected. [Recall that we say a space X is path-connected if for all $a, b \in X$, there exists a continuous function $f : [0, 1] \to X$ such that $f(0) = a$ and $f(1) = b$.]