Spring 2010 Topology Comprehensive Exam

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Do any 5 of the 7 problems below; they are worth 20 points each

1. Let d be a metric on $\mathbb{R}^n$ such that d is equivalent to the Euclidean metric. Let $U = \{(x_1, x_2, x_3, \ldots, x_n) : \sum_{j=1}^{n} x_j^2 < 1\}$ be the usual open ball of radius 1 centered at the origin. Show that U is bounded with respect to d. [Hint: Consider the closure of U.]

2. (a) Let τ_1, τ_2 be two topologies on some set X. Show that $\tau_1 \cap \tau_2$ is a topology on X.

(b) Give an example of two topologies τ_1, τ_2 on $\{a, b, c, d, e\}$ such that $\tau_1 \cup \tau_2$ is not a topology.

3. Let X be a topological space, and let Y be a Hausdorff space. Let f, g be two continuous functions from X to Y. Let $A = \{x \in X \mid f(x) = g(x)\}$. Show that A is closed in X.

4. Let n be a positive integer. Let $S^n = \{\mathbf{x} \in \mathbb{R}^n : ||\mathbf{x}|| = 1\}$, where $|| \cdot ||$ denotes the usual Euclidean norm on $\mathbb{R}^n$. Show that S^n is path connected.

5. In a topological space X, let $\overline{A}$ denote the closure of a subset A and let A' denote its set of limit (a.k.a. cluster or accumulation) points. Prove

(a) Prove $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) Show that if X is Hausdorff, then A' is closed.

6. (a) Let X be a Hausdorff space and let A be a compact subset. Prove A is closed.

(b) Let X compact topological space and let C be a closed subset. Prove C is compact.

7. (a) What does it mean for a topological space X to be locally connected?

(b) Give an example of a connected topological space X that is not locally connected. Clearly explain why X is connected, citing appropriate theorems; similarly, explain why the definition you provide in (a) fails to be satisfied by X.