Comprehensive Examination – Topology

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Do any five of the problems that follow. Each problem is worth 20 points. The set of positive integers is denoted by N, the set of rationals by Q, and the set of real numbers by R. The notation A^c means the complement of the set A with respect to an understood universal set. The notation $A \setminus B$ means $\{a : a \in A \text{ and } a \notin B\}$.

1. Let X, Y, and Z be topological spaces.

(a) Define the product topology on $Y \times Z$.

(b) Let $f : X \to Y$ and $g : X \to Z$ and define $h : X \to Y \times Z$ by $h(x) = (f(x), g(x))$. Use (a) and any property of the product topology (except the one you have to prove) to show that h is continuous if and only if f and g are continuous.

2. Let d_1 and d_2 be metrics on the same space X and let τ_1 and τ_2 be the corresponding topologies. Prove that the following are equivalent:

(a) Whenever $\{x_n\} \xrightarrow{d_1} x$ then $\{x_n\} \xrightarrow{d_2} x$ (that is whenever a sequence converges with respect to d_1 it also converges with respect to d_2, and to the same limit).

(b) $\tau_2 \subseteq \tau_1$.

3. (a) Define the notion of compact space.

(b) Prove that a compact subset of a metric space is closed and bounded.

4. Let X and Y be topological spaces, Y being Hausdorff. Let $f, g : X \to Y$ be continuous.

(a) Prove that the set $C = \{x \in X : f(x) = g(x)\}$ is closed in X.

(b) Let $A \subset X$ and assume that $f(a) = g(a)$ for all $a \in A$. Prove that $f(x) = g(x)$ for all $x \in \overline{A}$.

5. For each of the following pairs A, B of topological spaces determine whether they are homeomorphic. In each case, either display a homeomorphism or give a convincing reason why they cannot be homeomorphic.

(a) A is the open interval $(0,1)$ and $B = R$ (both with the usual topology).

(b) A is the open interval $(0,1)$ and $B = (0,1]$ (both with the usual topology).

(c) $A = R$ and $B = R^2$ (both with the usual topology).

(d) $A = Z$ and $B = Q$, both as subspaces of R.

(e) $A = R$ with the usual topology and $B = \{(x,y) \in R^2 : xy = 1,\ x > 0\}$ as a subspace of R^2.

6. Suppose $\gamma : [0,1] \to R^2$ is continuous, $\gamma(0) = (0,0)$ and $\gamma(0) = (2,3)$. Let $K = \{t \in [0,1] : \|\gamma(t)\| = 1\}$ (here $\|(x,y)\|$ is the usual norm on R^2, i.e. $\|(x,y)\| = \sqrt{x^2 + y^2}$). Show that K is closed and non empty.

7. Let A and B be subsets of a Hausdorff topological space X. Prove or disprove:

(a) $\overline{A \cup B} = \overline{A} \cup \overline{B}$.

(b) $(A \cup B)^\circ = A^\circ \cup B^\circ$. (Here A° means the interior of A.)