Comprehensive Examination – Topology

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Do five of the seven problems that follow. Each problem is worth 20 points. The notation A^c means the complement of the set A with respect to an understood universal set.

1. Let X be a Hausdorff space with more than one point. We call X *totally disconnected* if the components of X are singletons.

(a) Give an example of a totally disconnected space whose topology is not discrete. Provide some justification.

(b) Suppose X has a basis consisting of sets that are both open and closed. Prove that X is totally disconnected.

2. Let X and Y be topological spaces. Equip $X \times Y$ with the product topology. Suppose $A \subseteq X$ and $B \subseteq Y$.

(a) Show that $\overline{A \times B} = \overline{A} \times \overline{B}$.

(b) Show that $\text{int}(A \times B) = \text{int}_X(A) \times \text{int}_Y(B)$.

3. Let (X, d) be a metric space.

(a) Suppose (x_n) is a sequence in X convergent to x. Let $p \in X$ be arbitrary. Prove that $\lim_{n\to\infty} d(x_n, p) = d(x, p)$.

(b) Let A be a compact subset of X and suppose $p \in A^c$. Show that there exists $a_0 \in A$ with $d(p, a_0) = \inf_{a\in A} d(p, a)$.

4. Let X be a Hausdorff space. Prove that the following statements are equivalent:

(a) Whenever A is a nonempty closed subset of X and $p \in A^c$, there exist disjoint open sets V and W with $A \subset V$ and $p \in W$;

(b) For each $p \in X$ and each neighborhood U of p, there exists a neighborhood V of p with $\overline{V} \subset U$;

(c) Whenever A is a nonempty closed subset of X and $p \in A^c$, there exists a neighborhood V of p such that $\overline{V} \cap A = \emptyset$.

5. Let X and Y be Hausdorff spaces. For $f : X \to Y$, set $\Gamma(f) = \{(x, y) \in X \times Y : y = f(x)\}$. Prove or disprove each statement:

(a) If f is continuous, then $\Gamma(f)$ is a closed subset of $X \times Y$;

(b) If $\Gamma(f)$ is closed, then f is continuous.

6. Let (X, d) and (Y, ρ) be metric spaces, and let $f : X \to Y$.

(a) What does it mean for f to be globally continuous in terms of the open set structures of X and Y?

(b) What is the usual ε-δ criterion for continuity at each point of X that is equivalent to the open-set definition you gave in (a)?

(c) Show that the two definitons are equivalent.

7. Let τ be a compact Hausdorff topology on X, and let τ^* be a strictly larger topology on X, that is, $\tau \subset \tau^*$ but $\tau \neq \tau^*$. Show that τ^* cannot be compact.