Comprehensive Examination – Topology

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Do five of the problems that follow, ***including the first one***. Each problem is worth 20 points. The set of positive integers is denoted by N, the set of rationals by Q, and the set of real numbers by R. The notation A^c means the complement of the set A with respect to an understood universal set. The notation $A \setminus B$ means $\{a : a \in A \text{ and } a \notin B\}$.

1. Explain carefully 5 of the following concepts:

(a) Connected component of a topological space.

(b) Uniformly continuous function between two metric spaces.

(c) Totally bounded metric space.

(d) Product topology.

(e) Compact topological space.

(f) Basis for a topology.

2. Let X be a Hausdorff space and let $f, g : X \to R$ be continuous functions. Prove that each function below is continuous.

(a) $s : X \to R$, where $s(x) = f(x) + g(x)$;

(b) $a : X \to R$, where $a(x) = |f(x)|$;

(c) $m : X \to R$, where $m(x) = \max\{f(x), g(x)\}$.

3. Recall that a subset E of a metric space $\langle X, d\rangle$ is called *bounded* provided E is contained in some ball.

(a) Show that E is bounded if and only if $\sup\{d(x,y); x, y \in E\} < \infty$.

(b) Show that if E is compact then E is bounded.

(c) Show that if E is bounded then its closure $\overline{\mathrm{E}}$ is also bounded.

4. Let $N = \{1, 2, 3, \ldots\}$ be the set of positive integers and for any $i \in N$ let $X_i = \{0, 1\}$ be a two point space with the discrete topology. On $X = \prod_i X_i$ consider the product topology.

(a) Show that an open subset of X cannot consist of exactly one point.

(b) Show that a non empty connected subset of X consists of exactly one point.

5. Let X be a topological space and $\{A_i\}_{i \in N}$ be a sequence of connected subspaces of X.

(a) Show that if $\bigcap_{i \in N} A_i \neq \emptyset$ then $\bigcup_{i \in N} A_i$ is connected.

(b) Show that if $A_i \cap A_{i+1} \neq \emptyset$, $i = 1, 2, \ldots$, then $\bigcup_{i \in N} A_i$ is connected.

6. Let X, Y be metric spaces, X being compact. Prove that a continuous mapping $f : X \to Y$ is uniformly continuous.

7. Prove that a Hausdorff compact space is normal.