Comprehensive Examination – Topology

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Do any five of the problems that follow. Each problem is worth 20 points. The set of positive integers is denoted by N, the set of rationals by Q, and the set of real numbers by R. The notation A^c means the complement of the set A with respect to an understood universal set. The notation $A \setminus B$ means $\{a : a \in A \text{ and } a \notin B\}$.

1. Recall that a set is countable if it is either finite or there exists a bijection between the set and N. On R consider the following family of subsets

$$\sigma = \{D \subseteq R : D = \emptyset \text{ or } D^c \text{ is countable}\}$$

(a) Prove that σ is a topology on R.

(b) Show that R equipped with this topology a is connected.

(c) Show that σ is neither larger nor smaller than the usual topology τ on R.

2. (a) Define the notion of *closure* $\overline{A}$ of a subset A of a topological space (X, τ).

(b) Define the notion of *boundary* $\text{bd}(A)$ of a subset A of a topological space (X, τ).

(c) Prove, using only your definitions, that $A^c \in \tau$ if and only if $\text{bd}(A) \subseteq A$. **You may not cite theorems you have learned about closed sets!**

(d) Prove, using only your definitions, that $A \in \tau$ if and only if $bd(A) = \overline{A} - A$.

3. For each of the following pairs of topological spaces, either (1) present a homeomorphism between them, or (2) give a convincing reason why they cannot be homeomorphic. If you choose to present a homeomorphism, you need not prove that it is bicontinuous. Simply write down the defining formula.

(a) R with the usual topology, and $(0, \infty)$ as a subspace of R;

(b) R with the usual topology, and Q as a subspace of R;

(c) $[0, 1]$ as a subspace of R and the unit circle as a subspace of R^2, where both the line and the plane are equipped with the usual topology.

4. A real sequence $(x_n) = x_1, x_2, x_3, \ldots$ is called bounded if there exists $B > 0$ such that $|x_n| < B$ for each positive integer n. Let X denote the set of all bounded real sequences (x_n).

(a) Suppose $(x_n), (y_n)$ are bounded sequences. Explain why $\sup\{|x_n - y_n| : n \in N\}$ is a nonnegative real number.

(b) Show that $d : X \times X \to [0, oo)$ defined by $d((x_n), (y_n)) = \sup\{|x_n - y_n| : n \in N\}$ defines a metric on X.

(c) Prove that $E = \{(x_n) \in X : x_l > x_2\}$ is an open set in X equipped with this metric.

5. Recall that in a Hausdorff space, singleton subsets are closed sets.

(a) Using a famous theorem in topology, explain why each normal Hausdorff space is completely regular.

(b) Prove that each Hausdorff completely regular space is regular.

(c) Prove that each compact Hausdoiff space is regular.

6. Let $\{(X_i, \tau_i) : i \in I\}$ be a family of topological spaces.

(a) What is meant by the product topology τ on $\Pi_{i \in I} X_i$?

(b) Suppose (W, σ) is another topological space, and $f : W \to \Pi_{i \in I} X_i$. Prove that f is continuous if and only if for each i the coordinate function $\pi_i \circ f$ from W to X_i is continuous.

7. (a) Let (X, τ) be a topological space, and give $\{0, 1\}$ the discrete topology. Show that X is connected if and only if each continuous function from X to $\{0, 1\}$ is constant.

(b) Suppose $C_1, C_2, C_3, \ldots$ is a sequence of connected subsets of a topological space $(Y, \tau))$ such that for all indices n we have $C_n \cap C_{n+1} \neq \emptyset$. Prove that $\bigcup_{n=1}^{\infty} C_n$ is connected as a subspace of Y.