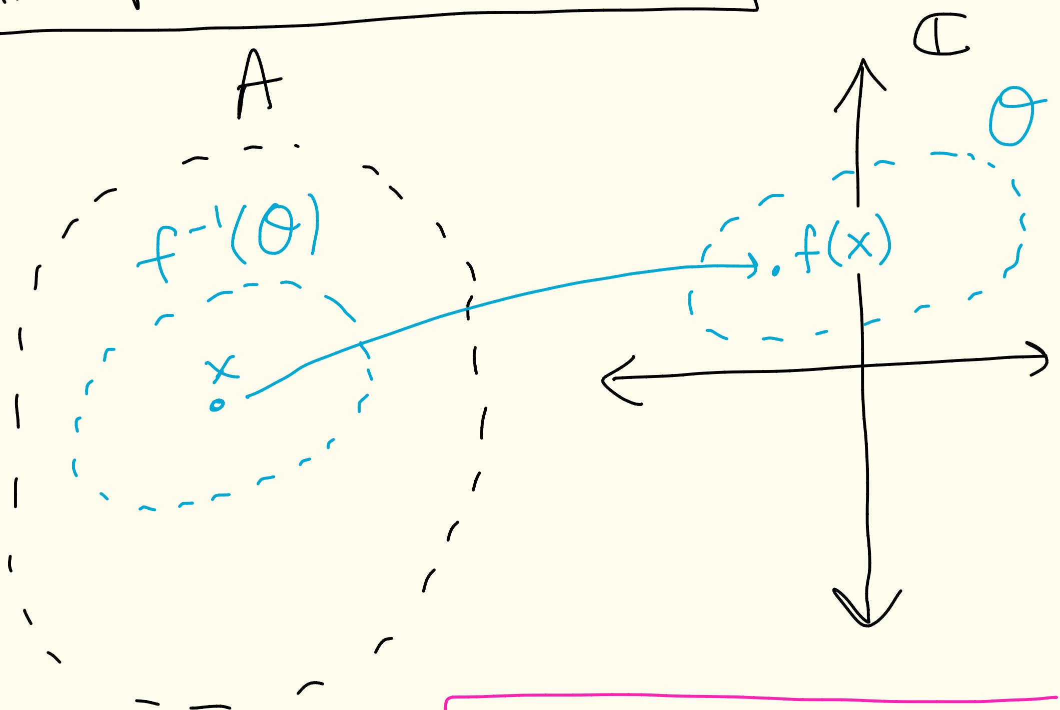
TOPIC 9 –
Some further results

4680 Result:

If $f: A \to \mathbb{C}$ is continuous on an open set A and $\mathcal{O} \subseteq \mathbb{C}$ is an open set, then

$$f^{-1}(\mathcal{O}) = \{ x \in A \mid f(x) \in \mathcal{O} \}$$

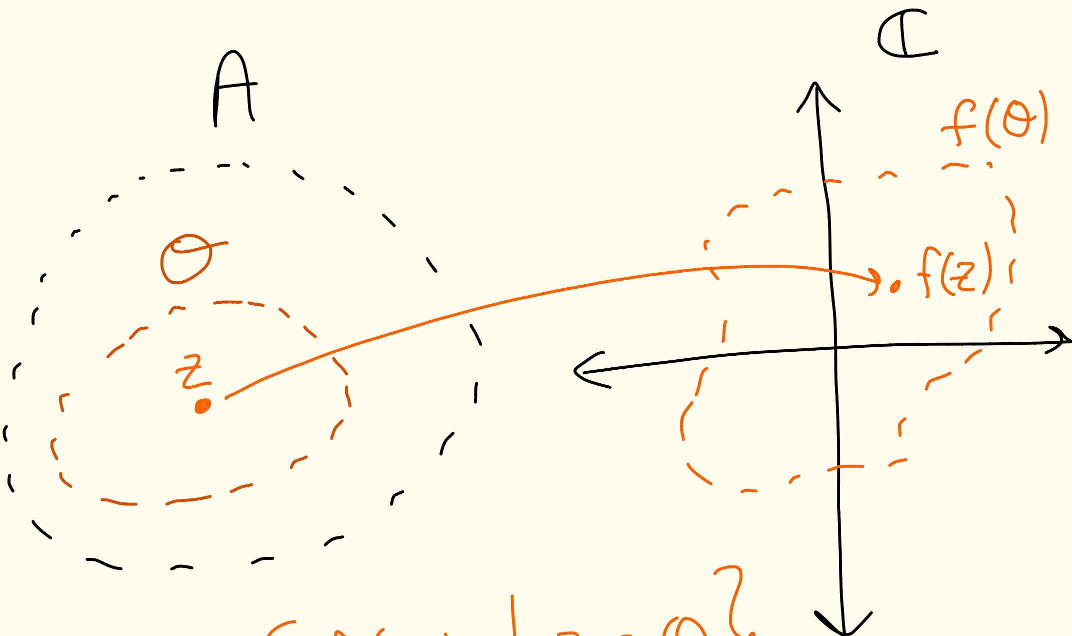
is open in $\mathbb{C}$



See Topic 9 supplement in notes online for proof

Open Mapping Theorem:

Let $f : A \to \mathbb{C}$ be an analytic function on a domain A.
If f is not constant on A, then $f(\mathcal{O})$ is open for all open sets $\mathcal{O} \subseteq A$.

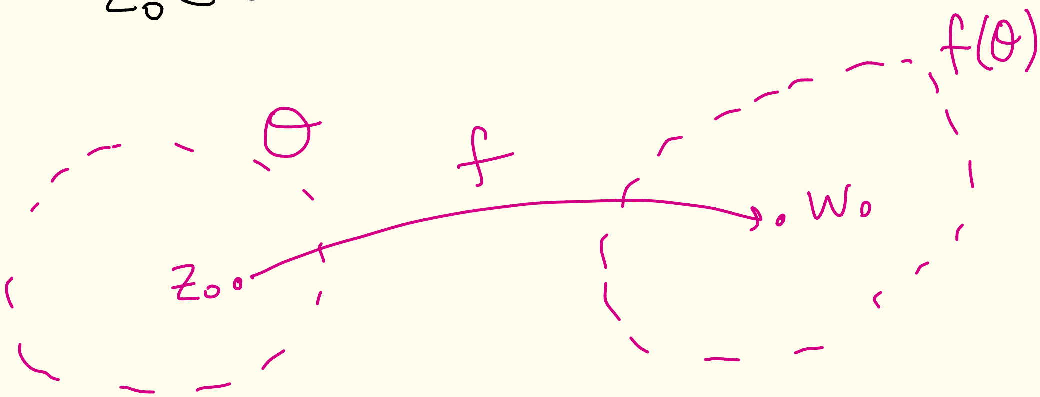


$$f(\mathcal{O}) = \{ f(z) \mid z \in \mathcal{O} \}$$

proof:

Let $\mathcal{O} \subseteq A$ be an open set.
We want to show that $f(\mathcal{O})$ is open.
Pick some $w_0 \in f(\mathcal{O})$.
We will show that w_0 is an interior point of $f(\mathcal{O})$.
This will complete the proof since w_0 was arbitrary.

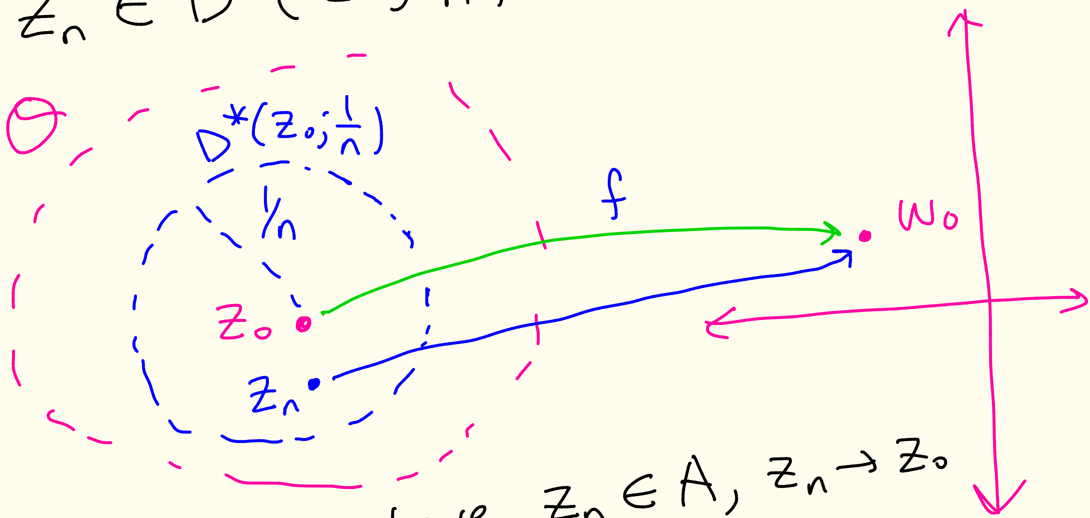
Since $w_0 \in f(\mathcal{O})$, there exists $z_0 \in \mathcal{O}$ with $f(z_0) = w_0$.



Claim: There exists $r > 0$ where $f(z) \neq w_0$ for all $z \in D^*(z_0; r)$

pf of claim:

If this claim was false then for each $n \geq 1$ there would exist $z_n \in D^*(z_0; \frac{1}{n})$ with $f(z_n) = w_0$.

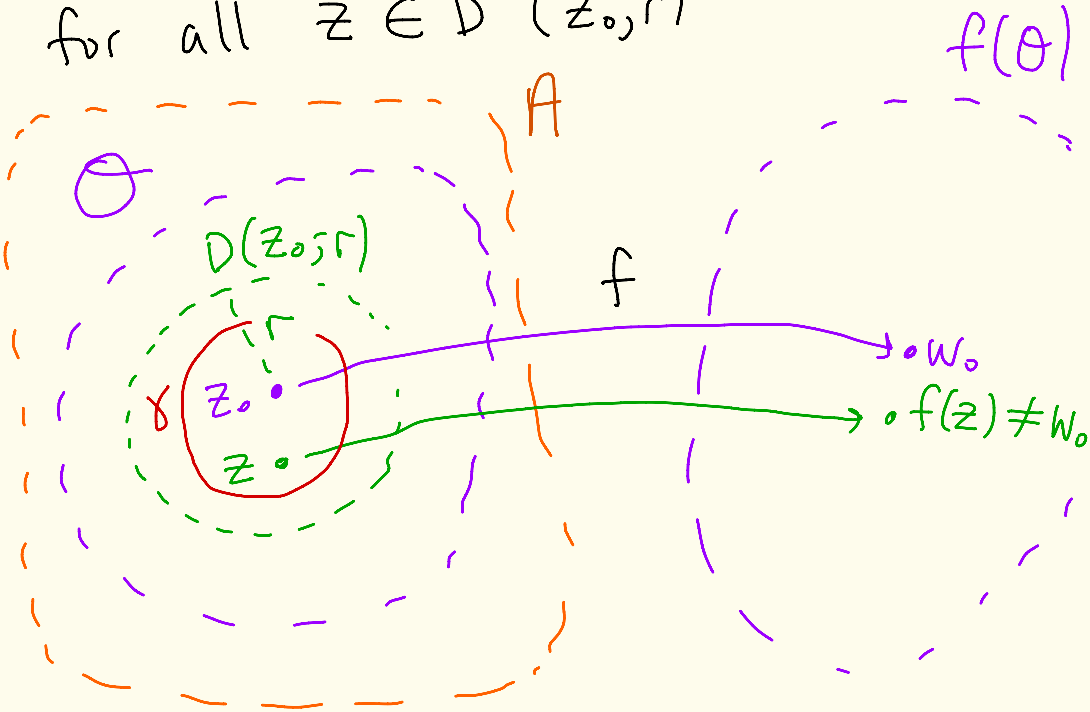


Then we would have $z_n \in A$, $z_n \to z_0$ and $f(z_n) = w_0$ for all $n \geq 1$.

But then by the identity theorem, f would be the constant function $f(z) = w_0$ for all $z \in A$, which isn't true

$\boxed{\text{claim}}$

Thus there exists $r > 0$ such that $D(z_0; r) \subseteq \mathcal{O}$ and $f(z) \neq w_0$ for all $z \in D^*(z_0; r)$



Let $F(z) = f(z) - w_0$.
So, F has an isolated zero at z_0.
Let γ be a circle inside of $D^*(z_0; r)$ surrounding z_0.

Then, $F(z) \neq 0$ in $D^*(z_0; r)$.

So, $|F(z)| > 0$ for all z on γ.

Because $F(z)$ is continuous on the compact set γ, F attains a minimum on γ.

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So, there exists $\rho > 0$ where

$$\underbrace{|F(z)|}_{|f(z) - w_0|} > \rho \quad \text{for all } z \text{ on } \gamma.$$

We will now show that

$$D(w_0; \rho) \subseteq f(\Theta).$$

This will show w_0 is an interior point of $f(\Theta)$ and hence $f(\Theta)$ is open.

Pick some $w_1 \in D(w_0; \rho)$

Let $H(z) = w_0 - w_1$ be a constant function on A.

Note that $|w_0 - w_1| < \rho$

Thus if z is on γ we have

$$|H(z)| = |w_0 - w_1| < \rho < |f(z) - w_0| = |F(z)|$$

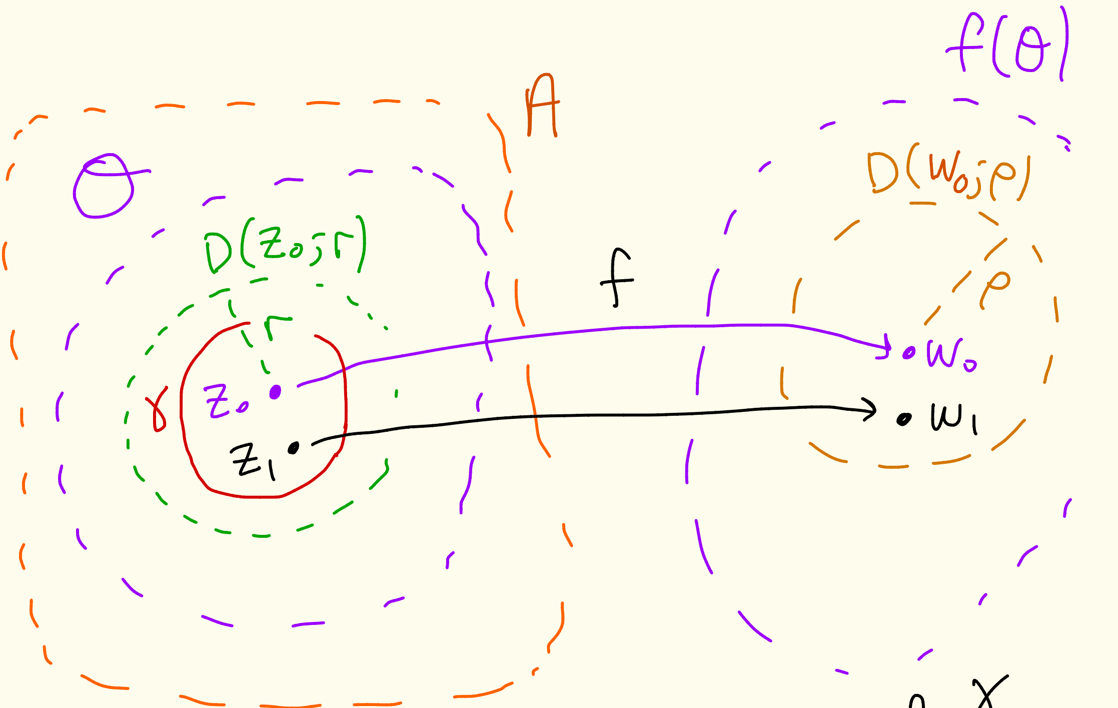
By Rouché's thm,

$$F(z) = f(z) - w_0$$

and

$$F(z) + H(z) = f(z) - w_0 + w_0 - w_1 = f(z) - w_1$$

have the same number of zeros inside of γ. We know $F(z)$ has the zero z_0 inside of γ. So, $F(z) + H(z) = f(z) - w_1$ also has a zero inside γ.



So there exists z_1 inside of γ, and hence $z_1 \in D^*(z_0;r)$, with $f(z_1) - w_1 = 0$ [ie $f(z_1) = w_1$].

Since $z_1 \in \mathcal{O}$, this shows that
$$w_1 \in f(\mathcal{O}).$$

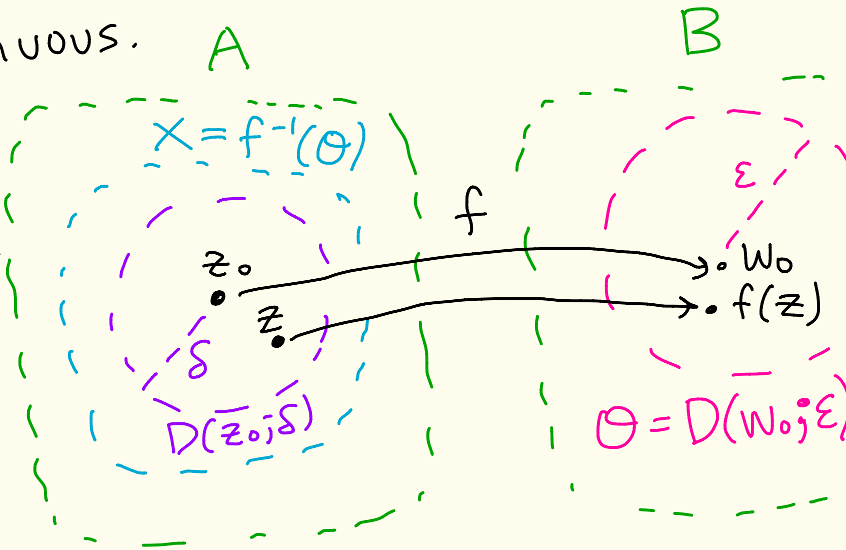
Hence, $D(w_0;\rho) \subseteq f(\mathcal{O})$.

Ie w_0 is an interior point of $f(\mathcal{O})$ ▨

Lemma: Let $f: A \to B$ where A and B be open subsets of $\mathbb{C}$. If $f^{-1}(\Theta)$ is open for every open set $\Theta \subseteq B$, then f is continuous.

pf:

Let $z_0 \in A$. Let $w_0 = f(z_0)$. Let's show f is continuous at z_0.



Let $\varepsilon > 0$. Let $\Theta = D(w_0;\varepsilon)$.

Then, $X = f^{-1}(\Theta) = \{z \in A \mid f(z) \in \Theta\}$ is open by our assumption on f.

Since $z_0 \in X$ and X is open, z_0 is an interior point of X.

So, $\exists \delta > 0$ where $D(z_0;\delta) \subseteq X$

Since $D(z_0;\delta) \subseteq X$ and $X = f^{-1}(\Theta)$ we know that if $z \in D(z_0;\delta)$ then $f(z) \in \Theta$.

Thus, if $|z - z_0| < \delta$, then

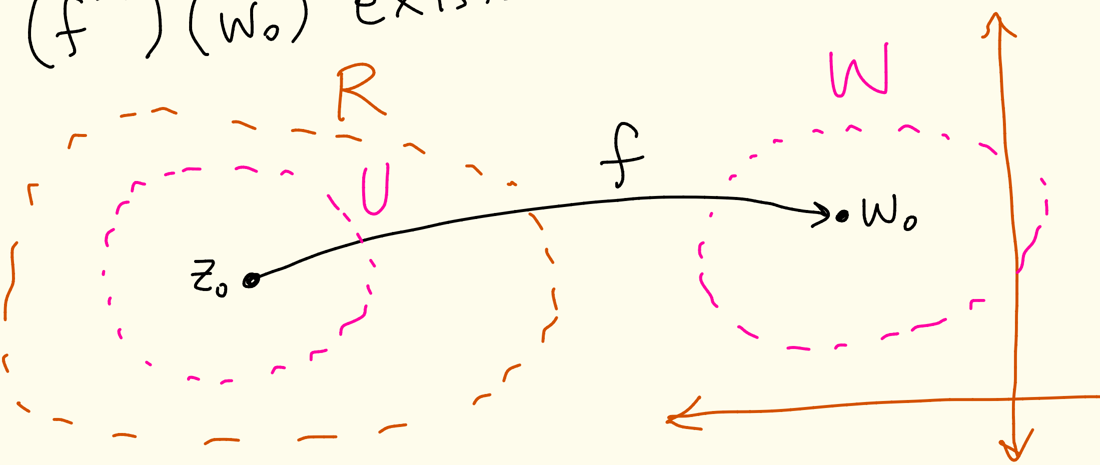
$$|f(z) - \underbrace{f(z_0)}_{w_0}| < \varepsilon$$

$\Theta = D(w_0;\varepsilon)$

So, f is continuous at z_0. ∎

Inverse function theorem

Let $f: R \to \mathbb{C}$ be analytic on a domain/region (open & connected). Let $z_0 \in R$. Suppose $f'(z_0) \neq 0$.
Then there exist open sets U and W in $\mathbb{C}$ such that $z_0 \in U \subseteq R$ and f maps U onto W bijectively.
So, $f^{-1}: W \to U$ exists.
Moreover, if $w_0 = f(z_0)$, then $(f^{-1})'(w_0)$ exists and $(f^{-1})'(w_0) = \frac{1}{f'(z_0)}$



proof:

Since f is analytic at z_0, there exists an $\varepsilon > 0$ where

$$f(z) = \underbrace{w_0}_{f(z_0)} + f'(z_0)(z-z_0) + \frac{f''(z_0)}{2!}(z-z_0)^2 + \dots$$

for all $z \in D(z_0;\varepsilon)$ and $D(z_0;\varepsilon) \subseteq R$

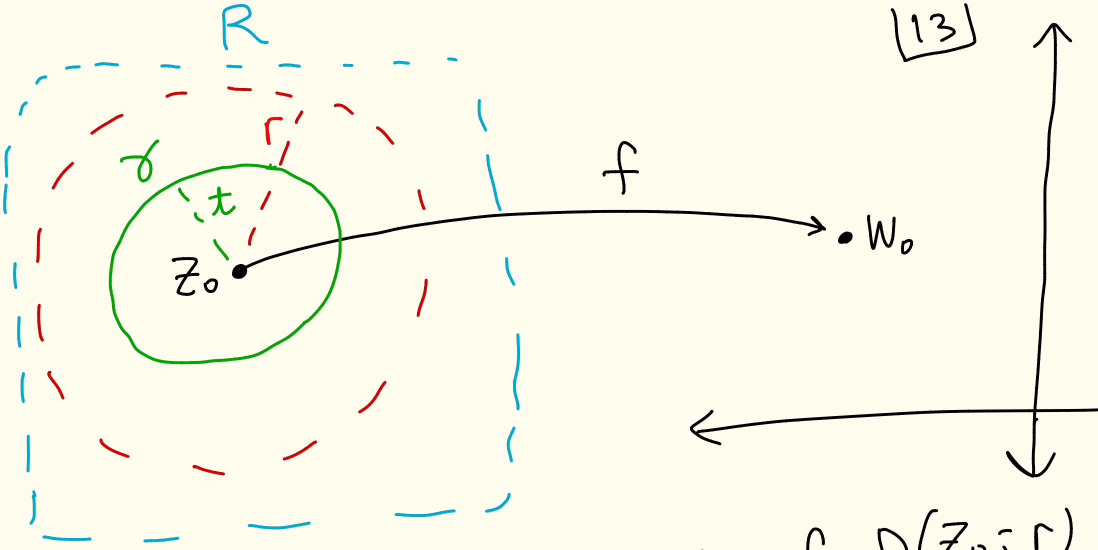
Thus,

$$f(z) - w_0 = (z-z_0)\left[\underbrace{f'(z_0)}_{\text{not } 0 \text{ by assumption}} + \frac{f''(z_0)}{2!}(z-z_0) + \dots\right]$$

So, $f(z) - w_0$ has a zero of order $\underline{1}$ at z_0.

Since the power series on the right does not have all zero coefficients we must have an isolated zero at z_0 [HW 3 #7].

Thus there exists $r > 0$ with $r < \varepsilon$ where $f(z) - w_0 \neq 0$ for all $z \in D^*(z_0;r)$

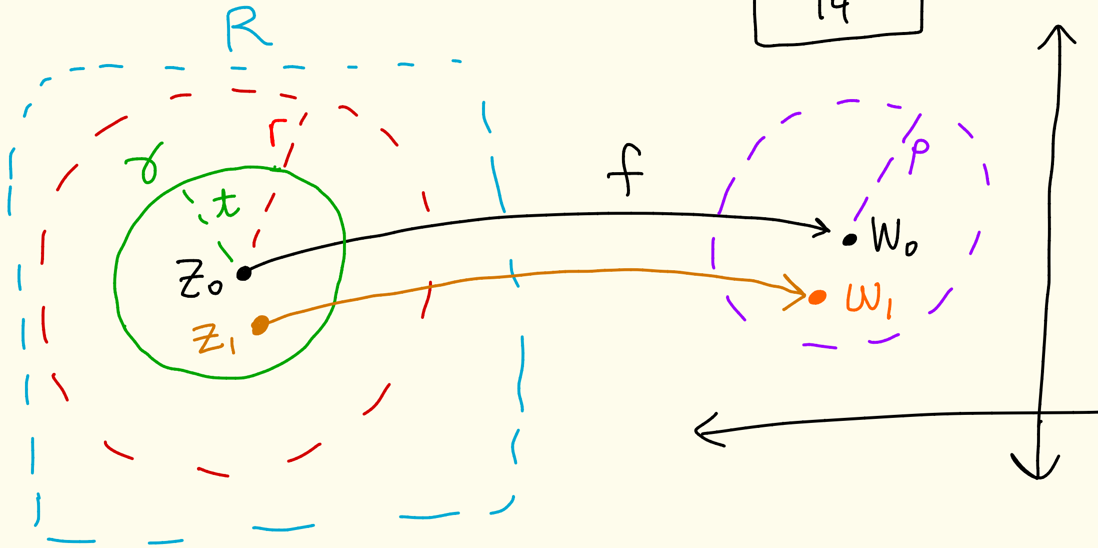


Let γ be a circle inside of $D(z_0; r)$ of radius $t < r$ centered at z_0

Because $\gamma \subseteq D(z_0; r)$ we have $f(z) - w_0 \neq 0$ for all z on γ.

Since γ is compact and $f(z) - w_0$ is continuous on γ, we know $|f(z) - w_0|$ achieves some minimum value on γ.

So, there exists $\rho > 0$ where $|f(z) - w_0| > \rho$ for all z on γ.



If $\underbrace{0<|w_0-w_1|<\rho}_{w_1 \in D(w_0;\rho) \text{ and } w_1 \neq w_0}$, then

$|w_0-w_1|<\rho<|f(z)-w_0|$ $\forall z$ on γ.

By Rouches theorem, $f(z)-w_0$ and $(w_0-w_1)+(f(z)-w_0)=f(z)-w_1$ have the same number of zeros inside of γ. We know $f(z)-w_0$ has exactly one zero inside of γ (counting multiplicity). Thus, there exists a unique z_1 inside γ where $f(z_1)-w_1=0$.

Let $W = D(w_0; \rho)$.

and $U = f^{-1}(W) \subseteq D(z_0; t)$

Since f is continuous and W is open, $U = f^{-1}(W)$ is open.

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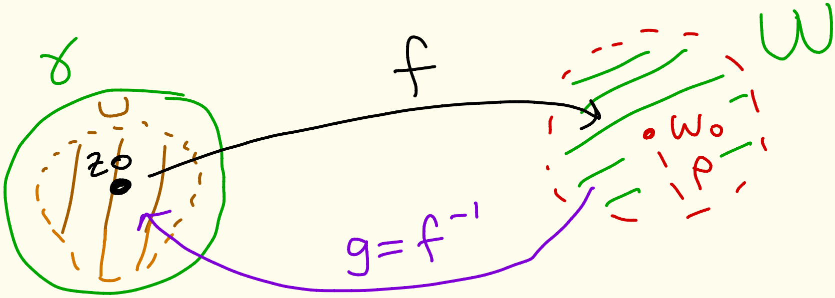
We just showed that f maps U onto W in a one-to-one way.



Let $g = f^{-1} : W \longrightarrow U$.

By the open mapping theorem for every open set $\mathcal{O} \subseteq U$ we have that $f(\mathcal{O})$ is open and hence $(f^{-1})^{-1}(\mathcal{O})$ is open. By the Lemma, f^{-1} is continuous on W.

Thus,

$$(f^{-1})'(w_0) = \lim_{w \to w_0} \frac{f^{-1}(w) - f^{-1}(w_0)}{w - w_0}$$

$$= \lim_{z \to z_0} \frac{z - z_0}{f(z) - f(z_0)}$$

$z = f^{-1}(w)$

$f(z) = w$

Since f^{-1} is continuous at w_0, as $w \to w_0$

$z \longrightarrow z_0$

where $z = f^{-1}(w)$

$z_0 = f^{-1}(w_0)$

$$= \lim_{z \to z_0} \frac{1}{\left(\frac{f(z) - f(z_0)}{z - z_0} \right)}$$

$$= \lim_{z \to z_0} \frac{1}{f'(z_0)}$$

$\blacksquare$