TOPIC 6 –
Applications of the Residue Theorem

HW 6 Topic – Applications of the Residue Theorem

Application I – Definite integrals involving sine and cosine

Recall:
$e^{i\theta} = \cos(\theta) + i\sin(\theta)$

__Ex:__ We will calculate

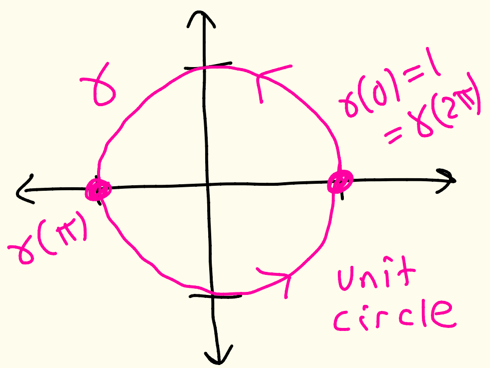
$$\int_0^{2\pi} \frac{d\theta}{5-4\cos(\theta)}$$

Let $z = e^{i\theta}$ where $0 \le \theta \le 2\pi$.

Let γ be the curve traced out by this equation, ie

$\gamma(\theta) = e^{i\theta}$

$0 \le \theta \le 2\pi$



Thus,

$$z = e^{i\theta}$$
$$dz = ie^{i\theta}\,d\theta$$
$$d\theta = \frac{dz}{ie^{i\theta}} = \frac{dz}{iz}$$

$$\boxed{e^{-i\theta} = \frac{1}{e^{i\theta}} = \frac{1}{z}}$$

$$\cos(\theta) = \frac{e^{i\theta} + e^{-i\theta}}{2} = \frac{z + \frac{1}{z}}{2}$$

Therefore,

$$\int_0^{2\pi} \frac{d\theta}{5 - 4\cos(\theta)} = \int_\gamma \frac{\frac{1}{iz}\,dz}{5 - 4\left(\frac{z+\frac{1}{z}}{2}\right)}$$

Why?

$$\boxed{\int_\gamma f(z)\,dz = \int_a^b f(\gamma(\theta))\,\gamma'(\theta)\,d\theta}$$

$$\int_\gamma \frac{\frac{1}{iz}\,dz}{5 - 4\left(\frac{z+\frac{1}{z}}{2}\right)} = \int_0^{2\pi} \frac{\frac{1}{ie^{i\theta}}}{5 - 4\left(\frac{e^{i\theta} + \frac{1}{e^{i\theta}}}{2}\right)} \cdot ie^{i\theta}\,d\theta$$

$$\boxed{\begin{aligned}\gamma(\theta) &= e^{i\theta} \\ \gamma'(\theta) &= ie^{i\theta}\,d\theta\end{aligned}}$$

$$= \int_0^{2\pi} \frac{d\theta}{5 - 4\cos(\theta)}$$

We get

$$\int_\gamma \frac{\frac{1}{iz}\,dz}{5-4\left(\frac{z+\frac{1}{z}}{2}\right)} = \frac{1}{i}\int_\gamma \frac{1}{5-2z-\frac{2}{z}}\cdot\frac{1}{z}\,dz$$

$$= -i\int_\gamma \frac{dz}{5z-2z^2-2} = i\int_\gamma \frac{dz}{2z^2-5z+2}$$

$\frac{1}{i} = -i$

When is $2z^2-5z+2=0$?

It's when

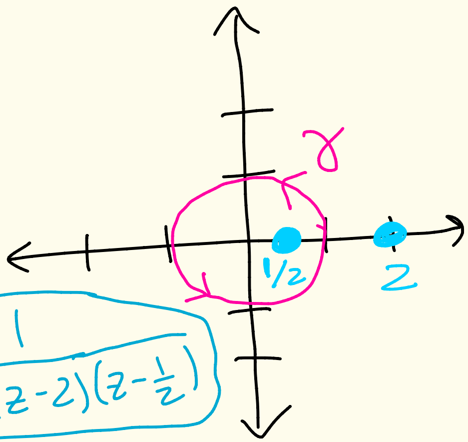
$$z = \frac{-(-5) \pm \sqrt{(-5)^2-4(2)(2)}}{2(2)}$$

$$= \frac{5\pm\sqrt{9}}{4} = \frac{5\pm 3}{4} = 2,\ \frac{1}{2}$$

Recall if $r_1, r_2 \in \mathbb{C}$ are the roots of $az^2+bz+c=0$, then
$$az^2+bz+c = a(z-r_1)(z-r_2)$$

Thus,
$$i\int_\gamma \frac{dz}{2z^2-5z+2} = i\int_\gamma \frac{dz}{2(z-2)(z-\frac{1}{2})}$$

We have singularities at $1/2$ and 2. But only $1/2$ is inside of γ.



$$f(z) = \frac{1}{2(z-2)(z-\frac{1}{2})}$$

So,
$$i\int_\gamma f(z)\,dz \underset{\text{Residue Thm}}{=} i\left[2\pi i \operatorname{Res}(f; \tfrac{1}{2})\right]$$

We have

$$f(z) = \frac{1}{2(z-2)(z-\frac{1}{2})} = \frac{\left[\frac{1}{2(z-2)}\right]}{(z-\frac{1}{2})}$$

$$= \boxed{\frac{\varphi(z)}{z-\frac{1}{2}}}$$

pole of order 1 at 1/2

where $\varphi(z) = \frac{1}{2(z-2)}$ and $\varphi\left(\frac{1}{2}\right) = \frac{1}{2(\frac{1}{2}-2)}$

$= -\frac{1}{3} \neq 0$

and φ is analytic at 1/2

Thus, f has a pole of order 1 at 1/2 and

$$\text{Res}\left(f; \frac{1}{2}\right) = \frac{\varphi^{(1-1)}\left(\frac{1}{2}\right)}{(1-1)!}$$

pole of order m

$$\frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$$

$$= \varphi\left(\frac{1}{2}\right) = -\frac{1}{3}$$

Thus,

$$\int_0^{2\pi} \frac{d\theta}{5-4\cos(\theta)} = i\left[2\pi i \,\mathrm{Res}\left(f;\tfrac{1}{2}\right)\right]$$

$$= -2\pi\left[-\frac{1}{3}\right] = \frac{2\pi}{3}$$

In general, suppose $R(x,y)$ is a rational function [ratio of polys] of x and y whose denominator does not vanish on the unit circle.

$R(x,y) \neq 0$ on unit circle $x^2+y^2=1$

To evaluate

$$\int_0^{2\pi} R(\cos(\theta), \sin(\theta))\, d\theta$$

$\sin(\theta) = \dfrac{e^{i\theta} - e^{-i\theta}}{2i}$

make the substitution $z = e^{i\theta}$ where $0 \le \theta \le 2\pi$ and use $\cos(\theta) = \frac{1}{2}\left(z + \frac{1}{z}\right)$ and $\sin(\theta) = \frac{1}{2i}\left(z - \frac{1}{z}\right)$ $d\theta = \dfrac{dz}{iz}$. Then use residue thm.

Application II – Improper Integrals

Recall that if $f(x)$ is a real-valued function for $x \in \mathbb{R}$ that is defined for $x \geq a$, then

$$\boxed{\int_a^\infty f(x)\,dx = \lim_{R\to\infty} \int_a^R f(x)\,dx}$$

Similary if $f(x)$ is defined for $x \leq a$, then

$$\boxed{\int_{-\infty}^a f(x)\,dx = \lim_{R\to\infty} \int_{-R}^a f(x)\,dx}$$

If $f(x)$ is defined for all $x \in \mathbb{R}$ then

$$\int_{-\infty}^{\infty} f(x)\,dx = \left[\lim_{R\to\infty} \int_{-R}^{a} f(x)\,dx\right] + \left[\lim_{R\to\infty} \int_{a}^{R} f(x)\,dx\right]$$

where a is any real number.

$\int_{-\infty}^{\infty} f(x)\,dx$ exists iff both integrals on the right-side exist.

<u>Fact:</u> Suppose $f: \mathbb{R} \to \mathbb{R}$ is even $[\text{ie } f(-x) = f(x) \text{ for all } x]$

If the <u>Cauchy principal value</u> of f $\left(\text{which is } \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx\right)$ exists, then $\int_a^\infty f(x)\,dx$ and $\int_{-\infty}^{a} f(x)\,dx$ exist for all a and

$$2\int_0^\infty f(x)\,dx = \int_{-\infty}^{\infty} f(x)\,dx = \lim_{R\to\infty} \int_{-R}^{R} f(x)\,dx$$

<u>proof:</u> Since f is even,

$$\int_{-R}^{0} f(x)\,dx = \int_0^R f(x)\,dx = \frac{1}{2}\int_{-R}^{R} f(x)\,dx$$

for all R. Now take limits as $R \to \infty$ ▨

Ex: We will calculate

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx$$

Let $f(x) = \dfrac{x^2}{x^6+1}$.

Then, $f(-x) = \dfrac{(-x)^2}{(-x)^6+1} = \dfrac{x^2}{x^6+1} = f(x)$

So, f is an even function on $\mathbb{R}$.

Thus, from last class,

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx = \frac{1}{2} \underbrace{\lim_{R\to\infty} \int_{-R}^{R} \frac{x^2}{x^6+1}\,dx}_{\text{Cauchy-principal value}}$$

Think of f in the complex plane

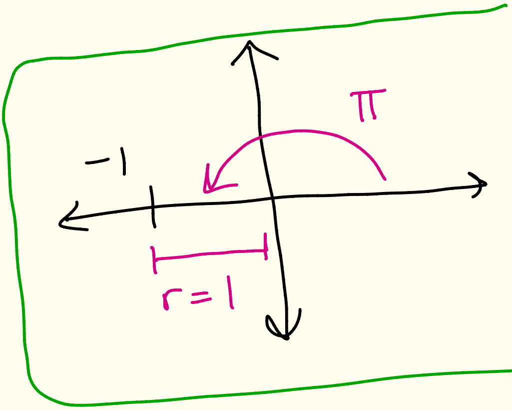
Ie, $f(z) = \dfrac{z^2}{z^6+1}$. The singularities

of f are when $z^6+1=0$.

So we need to solve

$$z^6 = -1 = 1 \cdot e^{i\pi}$$

roots are:



$$z_k = 1^{1/6} e^{\left(\frac{\pi}{6} + \frac{2\pi k}{6}\right)i}$$

$k = 0, 1, 2, 3, 4, 5$

$$z_0 = e^{\frac{\pi}{6}i}$$

$$z_1 = e^{\frac{\pi}{2}i}$$

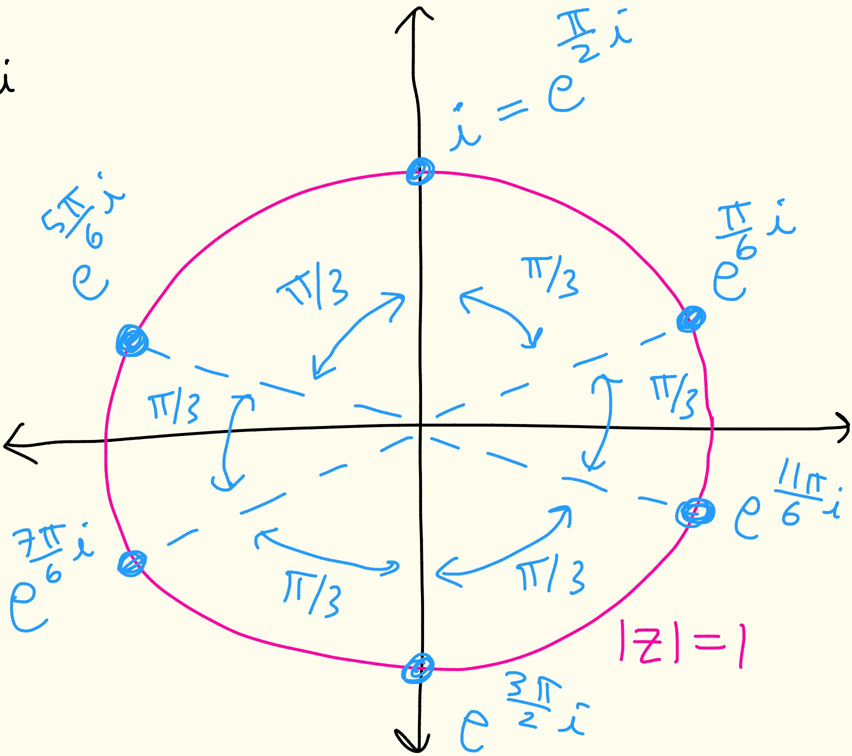
$$z_2 = e^{\frac{5\pi}{6}i}$$

$$z_3 = e^{\frac{7\pi}{6}i}$$

$$z_4 = e^{\frac{3\pi}{2}i}$$

$$z_5 = e^{\frac{11\pi}{6}i}$$

$$\boxed{\frac{2\pi}{6} = \frac{\pi}{3} \approx 60^\circ}$$

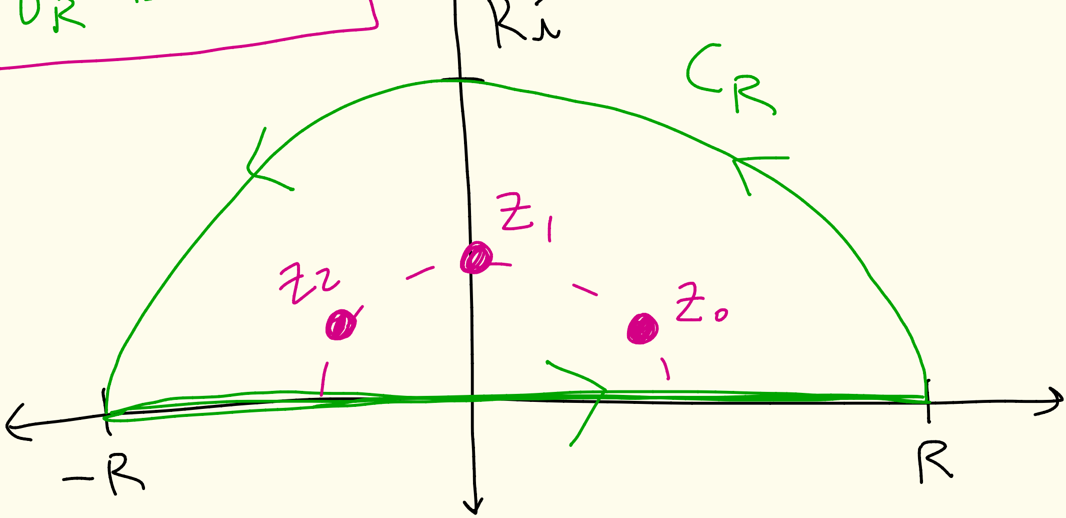


Given $R > 1$, let C_R be the upper half of the circle $|z| = R$ oriented counterclockwise.

Let γ_R be the closed curve formed by going along the x-axis from $-R$ to R and then going along C_R.

γ_R is 

$\boxed{R>1}$



Thus,

$$\int_{\gamma_R} f(z)\,dz = \int_{C_R} f(z)\,dz + \underbrace{\int_{-R}^{R} f(x)\,dx}_{\text{this part is a real integral}}$$

this part is a real integral
$z = x$ on real line from $-R$ to R

By the residue theorem,

$$\int_{\gamma_R} f(z)\,dz = 2\pi i \Big[\operatorname{Res}(f; z_0) + \operatorname{Res}(f; z_1) + \operatorname{Res}(f; z_2) \Big]$$

z_0, z_1, z_2 are all simple poles.

Why?

$$f(z) = \frac{z^2}{z^6+1} = \frac{g(z)}{h(z)}$$

$$\boxed{\begin{aligned} g(z) &= z^2 \\ h(z) &= z^6+1 \end{aligned}}$$

$$\left.\begin{aligned} g(z_k) &= z_k^2 \neq 0 \\ h(z_k) &= 0 \\ h'(z_k) &= 6(z_k)^5 \neq 0 \end{aligned}\right\} k=0,1,2$$

So, z_0, z_1, z_2 are simple poles

and

$$\mathrm{Res}(f; z_k) = \frac{g(z_k)}{h'(z_k)} = \frac{z_k^2}{6z_k^5}$$

$$= \frac{1}{6} \cdot \frac{1}{z_k^3}$$

So,

$$\mathrm{Res}(f; z_0) = \frac{1}{6} \cdot \frac{1}{z_0^3} = \frac{1}{6} \cdot \frac{1}{(e^{\pi i/6})^3}$$

$$= \frac{1}{6} \cdot \frac{1}{e^{\pi i/2}} = \frac{1}{6} \cdot \frac{1}{i}$$

$$\frac{1}{i} = -i$$

$$= \boxed{-\frac{1}{6} i}$$

$$\mathrm{Res}(f; z_1) = \frac{1}{6} \cdot \frac{1}{z_1^3} = \frac{1}{6} \cdot \frac{1}{i^3}$$

$$= \frac{1}{6} \cdot \frac{1}{-i} = \boxed{\frac{1}{6} i}$$

$$\mathrm{Res}(f; z_2) = \frac{1}{6} \cdot \frac{1}{z_2^3} = \frac{1}{6} \cdot \frac{1}{(e^{5\pi i/6})^3}$$

$$= \frac{1}{6} \cdot \frac{1}{e^{15\pi i/6}} = \frac{1}{6} \cdot \frac{1}{e^{3\pi i/6}}$$

$$= \frac{1}{6} \cdot \frac{1}{i} = \boxed{-\frac{1}{6} i}$$

Thus,

$$\int_{\gamma_R} f(z)\,dz = 2\pi i\left[-\frac{1}{6}i + \frac{1}{6}i - \frac{1}{6}i\right]$$

$$= 2\pi i\left[-\frac{1}{6}i\right] = \frac{\pi}{3}$$

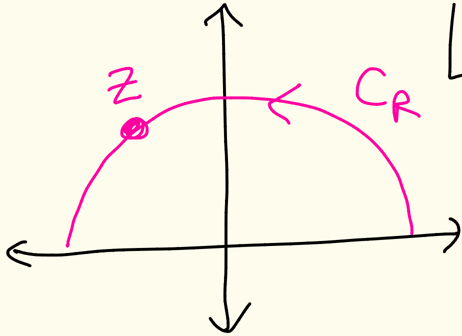
$i^2 = -1$

So,

$$\int_{-R}^{R} f(x)\,dx = \frac{\pi}{3} - \int_{C_R} f(z)\,dz$$

We want to let $R \to \infty$.

Suppose $R>1$.
Suppose $z \in C_R$.
Then, $|z|=R$



And so

$$|z^6+1| \geq \big|\,|z^6|-|1|\,\big| = \big|\,|z|^6-1\,\big|$$
$$= |R^6-1|$$
$$= R^6-1$$

$\boxed{|a+b| \geq \big|\,|a|-|b|\,\big|}$

$\boxed{\text{because } R>1}$

So, if $z \in C_R$, we have

$$|f(z)| = \left|\frac{z^2}{z^6+1}\right| = \frac{|z^2|}{|z^6+1|} = \frac{|z|^2}{|z^6+1|}$$
$$\leq \frac{R^2}{R^6-1}$$

Thus,

$$\left|\int_{C_R} f(z)\,dz\right| \leq \frac{R^2}{R^6-1} \cdot \underbrace{\pi R}_{\text{length of } C_R} = \frac{\pi R^3}{R^6-1}$$

So,

$$\left|\int_{C_R} f(z)\,dz\right| \leq \pi\left(\frac{\frac{1}{R^3}}{1-\frac{1}{R^3}}\right) \longrightarrow \pi\,\frac{0}{1-0} = 0$$

as $R\to\infty$

Thus,

$$\lim_{R\to\infty}\int_{-R}^{R} f(x)\,dx = \frac{\pi}{3} - \underbrace{\lim_{R\to\infty}\int_{C_R} f(z)\,dz}_{0}$$

So,

$$\int_0^\infty \frac{x^2}{x^6+1}\,dx = \frac{1}{2}\lim_{R\to\infty}\int_{-R}^{R}\frac{x^2}{x^6+1}\,dx$$

$$= \frac{1}{2}\left(\frac{\pi}{3}\right) = \frac{\pi}{6}. \quad \blacksquare$$

Improper Integrals involving sine and cosine

Sometimes if we want to evaluate

$$\int_{-\infty}^{\infty} f(x)\sin(ax)\,dx \quad \text{or} \quad \int_{-\infty}^{\infty} f(x)\cos(ax)\,dx$$

where $x \in \mathbb{R}$, $f(x) \in \mathbb{R}$, $a > 0$, $a \in \mathbb{R}$

then we can use

$$\int_{-R}^{R} f(x)\cos(ax)\,dx + i\int_{-R}^{R} f(x)\sin(ax)\,dx$$

$$= \int_{-R}^{R} f(x)\,e^{iax}\,dx$$

together with the fact that $|e^{iaz}| = e^{-ay}$ is bounded when $y \geq 0$

Ex: Let's show that

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+1)^2}\,dx = \frac{2\pi}{e^3}$$

$\cos(-\theta) = \cos(\theta)$

Note that $\dfrac{\cos(3(-x))}{((-x)^2+1)^2} = \dfrac{\cos(3x)}{(x^2+1)^2}$

So we have an even function, thus

$$\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+1)^2}\,dx = \underbrace{\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos(3x)}{(x^2+1)^2}\,dx}_{\text{Cauchy principal value}}$$

Let $f(z) = \dfrac{1}{(z^2+1)^2}$

If $z = x \in \mathbb{R}$, $\operatorname{Re}\left(f(x)\,e^{i3x}\right) =$

$$= \operatorname{Re}\left(\frac{e^{i3x}}{(x^2+1)^2}\right) = \operatorname{Re}\left(\frac{\cos(3x) + i\sin(3x)}{(x^2+1)^2}\right)$$

$$= \frac{\cos(3x)}{(x^2+1)^2}$$

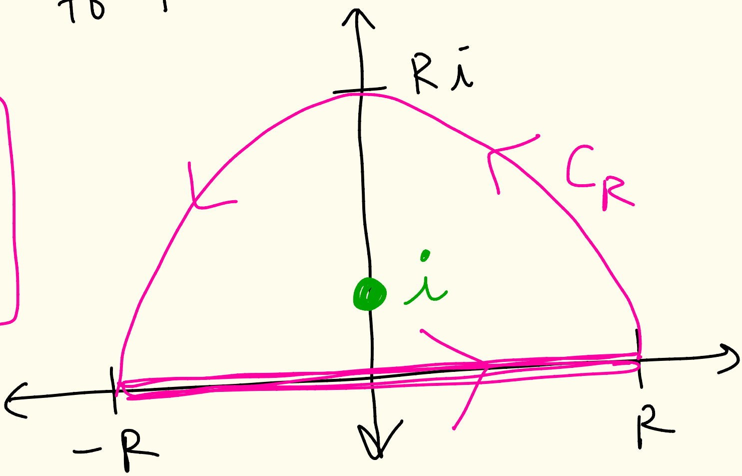
Note that

$$f(z)e^{i3z} = \frac{e^{i3z}}{(z^2+1)^2}$$

is analytic everywhere except where $z^2+1=0$ which is when $z = \pm i$.

Let $R>1$ and C_R be the top-half of the circle $|z|=R$ oriented counterclockwise.

Let γ_R be the curve from $-R$ to R and then along C_R.



γ_R is

When $R > 1$,

$$\int_{\gamma_R} f(z)e^{i3z}\,dz = 2\pi i \operatorname{Res}(f(z)e^{i3z}; i)$$

Note that

$$f(z)e^{i3z} = \frac{e^{i3z}}{(z^2+1)^2} = \frac{e^{i3z}}{[(z+i)(z-i)]^2}$$

$$= \frac{\dfrac{e^{i3z}}{(z+i)^2}}{(z-i)^2} = \frac{\varphi(z)}{(z-i)^2}$$

Note also that

$$\varphi(i) = \frac{e^{3i^2}}{(2i)^2} = \frac{e^{-3}}{-4} \neq 0$$

and φ is analytic at i.

So, we have a pole of order 2 at i and

$$\operatorname{Res}(f(z)e^{i3z}; i) = \frac{\varphi^{(2-1)}(i)}{(2-1)!}$$

$$= \varphi'(i)$$

We have that

$$\varphi'(z) = \frac{3ie^{i3z}(z+i)^2 - 2(z+i)e^{i3z}}{(z+i)^4}$$

Thus,

$$\varphi'(i) = \frac{3ie^{i3(i)}(2i)^2 - 2(2i)e^{i3(i)}}{(2i)^4}$$

$$= \frac{-12ie^{-3} - 4ie^{-3}}{16} = \frac{-i}{e^3}$$

So, $$\boxed{\int_{\gamma_R} f(z)e^{i3z}\,dz} = 2\pi i\left(\frac{-i}{e^3}\right) = \frac{2\pi}{e^3}$$

$\updownarrow$ equal

Thus,

$$\boxed{\underbrace{\int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2}\,dx}_{z=x \text{ on the real line}} + \int_{C_R} \frac{e^{i3z}}{(z^2+1)^2}\,dz} = \frac{2\pi}{e^3}$$

Take the real part of both sides and use the fact that

$$\int_{-R}^{R} \frac{e^{i3x}}{(x^2+1)^2}\,dx = \int_{-R}^{R} \frac{\cos(3x)}{(x^2+1)^2}\,dx + i\int_{-R}^{R} \frac{\sin(3x)}{(x^2+1)^2}\,dx$$

$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta), \quad \theta \in \mathbb{R}}$$

we get

$$\boxed{\int_{-R}^{R} \frac{\cos(3x)}{(x^2+1)^2}\,dx + \operatorname{Re}\left[\int_{C_R} \frac{e^{i3z}}{(z^2+1)^2}\,dz\right] = \frac{2\pi}{e^3} \quad (*)}$$

<u>Goal:</u> Show $\displaystyle\int_{C_R} \frac{e^{i3z}}{(z^2+1)^2}\,dz \to 0$

as $R \to \infty$

Let $z = x + iy$ be on C_R.

Then, $|z| = R$.

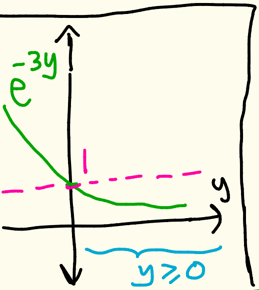
So we have

$$|z^2+1| \geq \big|\,|z^2| - |1|\,\big| = \big|\,|z|^2 - 1\,\big|$$
$$= |R^2 - 1| = R^2 - 1$$

(since $R>1$)

and

$$|e^{i3z}| = |e^{i3(x+iy)}| = \underbrace{|e^{i3x}|}_{|e^{i\theta}|=1,\ \theta\in\mathbb{R}}|e^{-3y}|$$



$$= e^{-3y} = \frac{1}{e^{3y}} \leq 1$$

$\boxed{y \geq 0}$

and

$$\left|\frac{e^{i3z}}{(z^2+1)^2}\right| = \frac{|e^{i3z}|}{|z^2+1|^2} \leq \frac{1}{(R^2-1)^2}$$

So,

$$\left| \int_{C_R} \frac{e^{i3z}}{(z^2+1)^2}\,dz \right| \leq \frac{1}{(R^2-1)^2} \cdot \underbrace{\pi R}_{\text{length of } C_R}$$

$$= \frac{\pi R}{R^4 - 2R^2 + 1} \to 0 \quad \text{as} \quad R \to \infty$$

Thus, taking $R \to \infty$ in $(*)$ and using the fact that

$$\lim_{R\to\infty} \int_{-R}^{R} \frac{\cos(3x)}{(x^2+1)^2}\,dx = \int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+1)^2}\,dx$$

(because the function is even)

we get that $\displaystyle\int_{-\infty}^{\infty} \frac{\cos(3x)}{(x^2+1)^2}\,dx = \frac{2\pi}{e^3}$ $\blacksquare$

Note: We needed to do the e^{i3z} substitution otherwise on C_R the integral wouldn't go to zero since

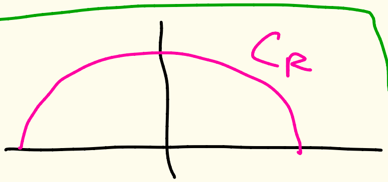
$$\cos(z) = \frac{e^{iz} + e^{-iz}}{2}$$

is unbounded in the upper-half plane. because

$$|e^{iz}| = e^{-y} \qquad |e^{-iz}| = e^{y}$$

goes to ∞ as y gets bigger

$0 \leq y \leq R$



Could also look at $z = iR$ on C_R

Then $\cos(z) = \cos(iR) = \dfrac{e^{i(iR)} + e^{-i(iR)}}{2}$

$$= \frac{\overset{\to 0}{e^{-R}} + \overset{\to \infty}{e^{R}}}{2} \longrightarrow \infty$$

as $R \to \infty$

Then if z is C_R you'd try

$$\left| \frac{\cos(z)}{(z^2+1)^2} \right| \leq \frac{e^R \text{ stuff}}{(R^2-1)^2} \not\to 0$$