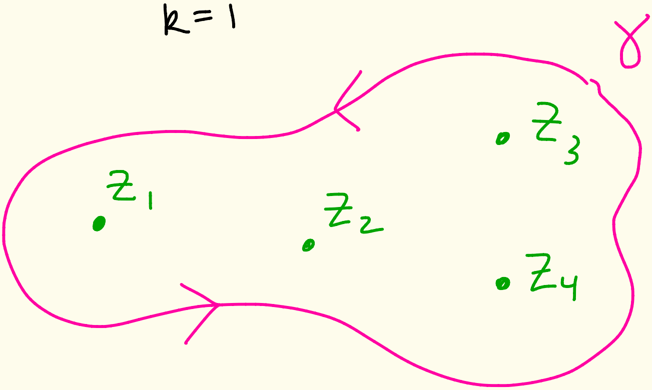
Topic 5 –
The Residue Theorem

Theorem: (Cauchy's Residue Thm)

Let γ be a simple, closed, piecewise smooth curve, oriented in the counterclockwise direction. If f is analytic inside and on γ except for a finite number of isolated singularities $z_1, z_2, \ldots, z_n$ of the function f that lie inside γ, then

$$\int_\gamma f(z)\,dz = 2\pi i \sum_{k=1}^{n} \mathrm{Res}(f; z_k)$$

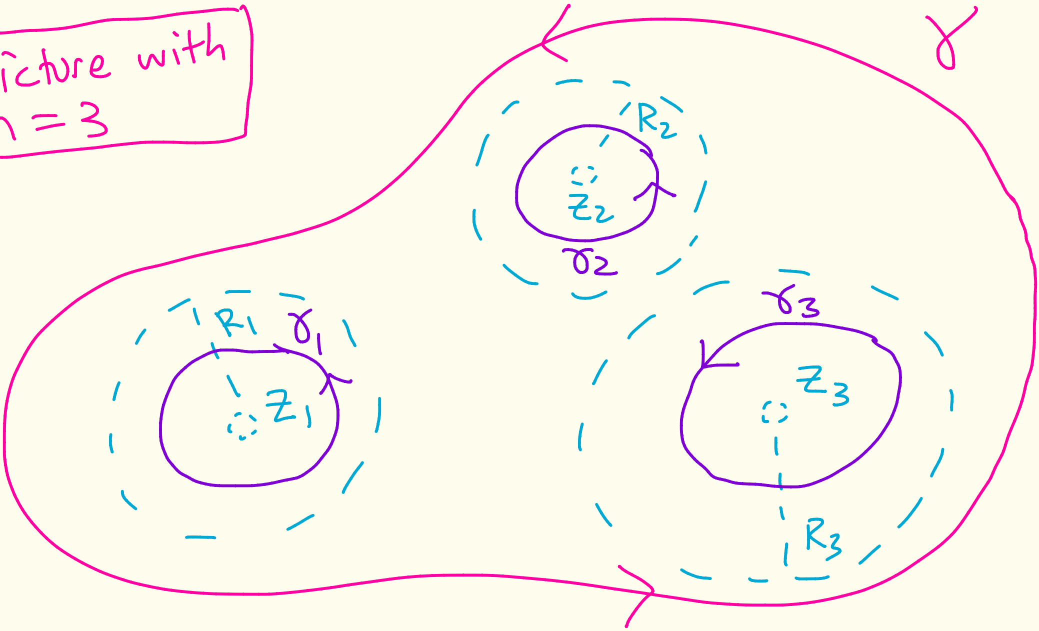
picture for $n=4$



proof:

For each z_k, there is a number $R_k > 0$ such that f is analytic on $D^*(z_k; R_k)$ and $D^*(z_k; R_k)$ lies inside γ. Pick each R_k so that none of the deleted neighborhoods of $z_1, z_2, \ldots, z_n$ overlap. Let γ_k be a counter-clockwise oriented circle centered at z_k and contained inside $D^*(z_k; R_k)$

picture with $n=3$



From 4680, since f is analytic on and in-between γ and $\gamma_1, \gamma_2, \ldots, \gamma_n$ we have that

$$\int_\gamma f(z)\,dz = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz$$

Let's take a look at $\int_{\gamma_k} f(z)\,dz$.

Inside of $D^*(z_k; R_k)$ we have a Laurent series expansion

$$f(z) = \cdots + \frac{b_2}{(z-z_k)^2} + \frac{b_1}{(z-z_k)} + a_0 + a_1(z-z_k) + \cdots$$

and $b_1 = \frac{1}{2\pi i} \int_{\gamma_k} f(z)\,dz$ $(\ell = 1)$

$\boxed{b_1 = \mathrm{Res}(f; z_k)}$

$$\boxed{\text{Recall: } b_\ell = \frac{1}{2\pi i} \int_{\gamma_k} f(z)\cdot(z-z_k)^{\ell-1}\,dz \quad \ell = 1, 2, 3, \ldots}$$

Thus,

$$\int_{\gamma} f(z) = \sum_{k=1}^{n} \int_{\gamma_k} f(z)\,dz$$

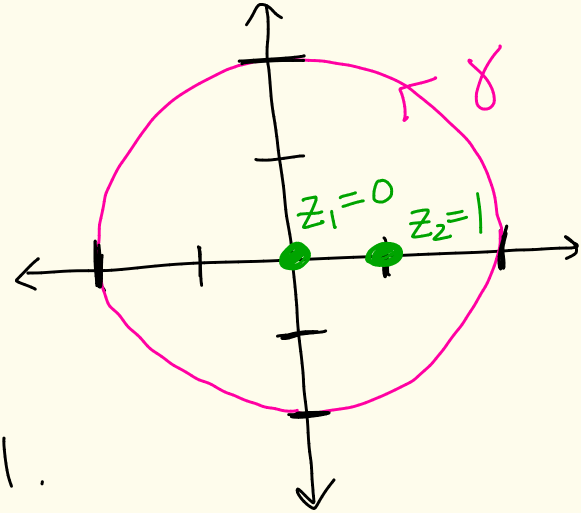
$$= 2\pi i \sum_{k=1}^{n} \operatorname{Res}(f; z_k)$$

$\blacksquare$

Ex:

Consider $\displaystyle\int_\gamma \frac{5z-2}{z(z-1)}\,dz$

where γ is the circle $|z|=2$ oriented counterclockwise.



Let

$$f(z) = \frac{5z-2}{z(z-1)}.$$

f has isolated singularities at $z_1=0$ and $z_2=1$.

By the residue theorem,

$$\int_\gamma \underbrace{\frac{5z-2}{z(z-1)}}_{f(z)}\,dz = 2\pi i\left[\operatorname{Res}(f;0)+\operatorname{Res}(f;1)\right]$$

$\boxed{\mathrm{Res}(f;0)}$

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{g(z)}{h(z)}$$

This is a method for simple poles

where $g(z) = 5z-2$

and $h(z) = z(z-1) = z^2 - z$

And,

$g(0) = 5(0) - 2 = -2 \neq 0$

$h(0) = 0^2 - 0 = 0$

$h'(0) = 2(0) - 1 = -1 \neq 0$

$h'(z) = 2z - 1$

So we have a simple pole at $z_1 = 0$

and

$$\mathrm{Res}(f;0) = \frac{g(0)}{h'(0)} = \frac{-2}{-1} = 2$$

$\boxed{Res(f;1)}$

$$f(z) = \frac{5z-2}{z(z-1)} = \frac{\left(\frac{5z-2}{z}\right)}{z-1} = \frac{\varphi(z)}{z-1}$$

where $\varphi(z) = \dfrac{5z-2}{z}$.

Then φ is analytic at $z_2 = 1$.

And $\varphi(1) = \dfrac{5(1)-2}{1} = 3 \neq 0$.

Thm from class

$$f(z) = \frac{\varphi(z)}{(z-z_0)^m}$$

φ is analytic at z_0

$\varphi(z_0) \neq 0$

Then,

f has a pole of order m at z_0

and

$$Res(f;z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}$$

Here we have a simple pole, ie of order 1.

And so,

$$Res(f;1) = \frac{\varphi^{(1-1)}(1)}{(1-1)!}$$

$$= \frac{\varphi(1)}{0!} = \varphi(1) = 3$$

So,

$$\int_{\gamma} \frac{5z-2}{z(z-1)}\, dz = 2\pi i \left[\mathrm{Res}(f;0) + \mathrm{Res}(f;1) \right]$$

$$= 2\pi i \left[2+3 \right]$$

$$= 10\pi i$$