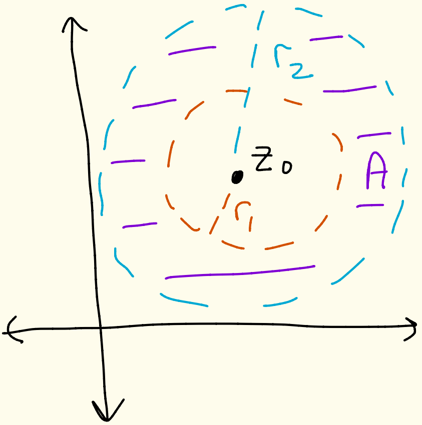
TOPIC 4 -
Laurent Series

<u>Theorem</u> (Laurent Expansion Theorem)

Let $0 \leq r_1 < r_2$ and $z_0 \in \mathbb{C}$.

Consider the annulus



$$A = \{ z \mid r_1 < |z - z_0| < r_2 \}$$

We allow $r_1 = 0$
or $r_2 = \infty$
or both.

Suppose that f is analytic in A.

Then we can write

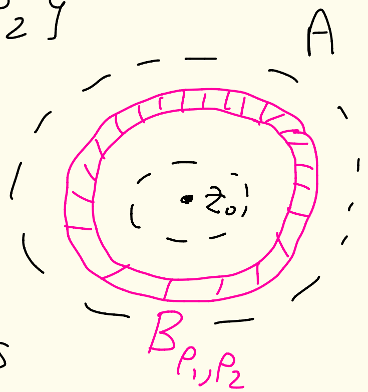
$$\boxed{f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n}$$

$$= \left[\cdots + \frac{b_2}{(z-z_0)^2} + \frac{b_1}{(z-z_0)} \right] + \left[a_0 + a_1(z-z_0) + \cdots \right]$$

Where both series on the right-hand side of the equation converge absolutely on A and uniformly in the sets of the form

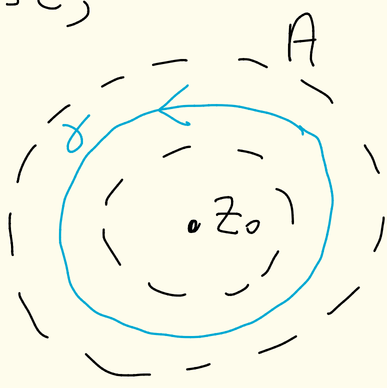
$$B_{\rho_1,\rho_2} = \{ z \mid \rho_1 \leq |z - z_0| \leq \rho_2 \}$$

where $r_1 < \rho_1 < \rho_2 < r_2$.



This series for f is called the <u>Laurent series of f centered at z_0 in the annulus A.</u>

If γ is a circle around z_0, oriented counterclockwise, with radius r where $r_1 < r < r_2$,



then

$$a_n = \frac{1}{2\pi i} \int_\gamma \frac{f(\zeta)}{(\zeta - z_0)^{n+1}} \, d\zeta$$ (3)

for $n = 0, 1, 2, \ldots$

and

$$b_n = \frac{1}{2\pi i} \int_\gamma f(\zeta) \cdot (\zeta - z_0)^{n-1} \, d\zeta$$

for $n = 1, 2, 3, \ldots$

Any pointwise convergent expansion of f of this form in A equals the Laurent expansion.

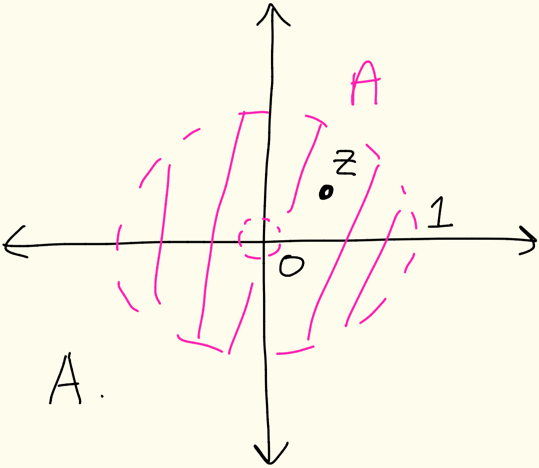
That is, the Laurent expansion in A is unique.

<u>proof:</u> Hoffman/Marsden book ▨

Ex: $f(z) = \dfrac{1}{z(z-1)}$

Let $A = \{ z \mid 0 < |z| < 1 \}$

f is analytic in A.



Let's find the Laurent series in A.

Let $z \in A$.

Then $0 < |z| < 1$.

So,

$$f(z) = \frac{1}{z(z-1)} = -\frac{1}{z}\left[\frac{1}{1-z}\right]$$

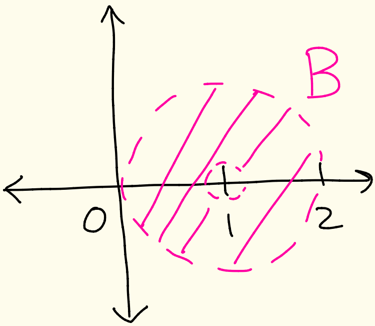
$$\underset{\boxed{|z|<1}}{=} -\frac{1}{z}\left[1 + z + z^2 + z^3 + z^4 + \cdots\right]$$

$$= \underbrace{\left[-\frac{1}{z}\right]}_{\sum_{n=1}^{\infty} \frac{b_n}{z^n}} + \underbrace{\left[-1 - z - z^2 - z^3 - \cdots\right]}_{\sum_{n=0}^{\infty} a_n z^n}$$

Ex: $f(z) = \dfrac{1}{z(z-1)}$

Let's expand f on

$B = \{ z \mid 0 < |z-1| < 1 \}$



f is analytic on B.

Let $z \in B$. Then,

$$\frac{1}{z(z-1)} = \frac{1}{(z-1)} \cdot \frac{1}{z} = \frac{1}{(z-1)} \cdot \frac{1}{1+(z-1)}$$

$$= \frac{1}{(z-1)} \cdot \frac{1}{1-(-(z-1))} =$$

⑥

$$= \frac{1}{(z-1)} \cdot \left[1 - (z-1) + [-(z-1)]^2 + [-(z-1)]^3 + \cdots \right]$$

$z \in B$

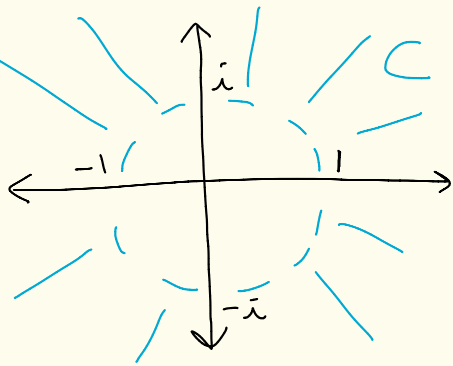
$|z-1| < 1$

$|-(z-1)| < 1$

$$= \frac{1}{z-1} - 1 + (z-1) - (z-1)^2 + (z-1)^3 - \cdots$$

Let

$C = \{ z \mid 1 < |z| < \infty \}$



f is analytic on C.

Let $z \in C$. Then,

$$\frac{1}{z(z-1)} = \frac{-1}{z} \cdot \frac{1}{\underbrace{1-z}_{\text{need } |z|<1}}$$

can't go further

Given $z \in C$ [ie $1 < |z|$]

we have

$$\frac{1}{z(z-1)} = \frac{1}{z} \cdot \frac{1}{z} \cdot \frac{1}{1-(1/z)}$$

$$= \frac{1}{z^2}\left[1 + \frac{1}{z} + \frac{1}{z^2} + \frac{1}{z^3} + \cdots\right]$$

$1 < |z|$

$\left|\frac{1}{z}\right| < 1$

$$= \frac{1}{z^2} + \frac{1}{z^3} + \frac{1}{z^4} + \frac{1}{z^5} + \cdots$$

$$= \cdots + \frac{1}{z^4} + \frac{1}{z^3} + \frac{1}{z^2}$$

Def: Let $z_0 \in \mathbb{C}$.

We say that z_0 is an <u>isolated singularity</u> of f if

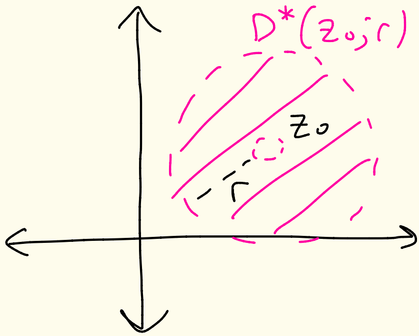
(1) f is not analytic at z_0

and (2) f is analytic on some deleted r-neighborhood

$$D^*(z_0; r) = \{ z \mid 0 < |z - z_0| < r \}$$

of z_0.

Note this shape  for def of isolated singularity ie remove only z_0



If this is the case, then

$$f(z) = \left[\cdots + \frac{b_n}{(z-z_0)^n} + \cdots + \frac{b_1}{(z-z_0)} \right] + \left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \right]$$

for $z \in D^*(z_0;r)$ where the above is the Laurent expansion of f in $D^*(z_0;r)$

Furthermore:

A) If all but a finite number of the b_n's are zero, then z_0 is called a pole of f. If k is the largest integer such that $b_k \neq 0$ then z_0 is called a pole of order k.

We refer to a pole of order 1 as a simple pole.

B) If an infinite number of the b_n's are non-zero, z_0 is called an essential singularity.

C) We call b_1 the residue of f at z_0 and write

$$\operatorname{Res}(f; z_0) = b_1$$

D If all the b_n's are zero we say that z_0 is a <u>removable singularity</u>.

In this case,

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \in D^*(z_0; r)$$

Define

$$\tilde{f}(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n \quad \forall z \in D(z_0; r)$$

Then, $\tilde{f}(z) = f(z) \quad \forall z \in D^*(z_0; r)$ but it is also defined at z_0, as

$$\tilde{f}(z_0) = a_0 + a_1(z_0 - z_0) + a_2(z_0 - z_0)^2 + \cdots$$
$$= a_0$$

$\tilde{f}$ is analytic on $D(z_0; r)$ because it's a power series. So, $\tilde{f}$ extends f to be an analytic function on $D(z_0; r)$

Ex: Let

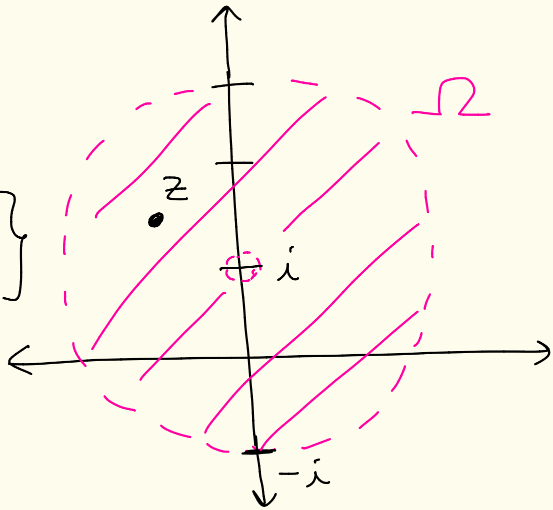
$$f(z) = \frac{z}{(z-i)(z^2+1)}$$

$z^2+1=0$
$(z+i)(z-i)=0$

f has isolated singularities at $z_0 = i, -i$.

Let

$$\Omega = \{ z \mid 0 < |z-i| < 2 \}$$



Let $z \in \Omega$

Note that

$$\frac{z}{(z-i)(z^2+1)} = \left(\frac{1}{z-i}\right)\left[\frac{z}{z^2+1}\right]$$

$$=$$

Use partial-fractions:

$$\underbrace{\frac{z}{(z+i)(z-i)}}_{\frac{z}{z^2+1}} = \frac{A}{z+i} + \frac{B}{z-i}$$

So, $z = A(z-i) + B(z+i)$

$\underline{z=i}$: $i = A(0) + B(2i) \to B = 1/2$

$\underline{z=-i}$: $-i = A(-2i) + B(0) \to A = 1/2$

Thus,

$$f(z) = \left(\frac{1}{z-i}\right)\left[\frac{1}{2}\cdot\frac{1}{z-i} + \frac{1}{2}\cdot\frac{1}{z+i}\right]$$

need to deal with

We have

$$\frac{1}{z+i} = \frac{1}{2i+z-i} = \frac{1}{2i}\left[\frac{1}{1-\left(\frac{-(z-i)}{2i}\right)}\right]$$

$$= \frac{1}{2i}\sum_{n=0}^{\infty}\left(\frac{-(z-i)}{2i}\right)^n = \sum_{n=0}^{\infty}\frac{(-1)^n(z-i)^n}{(2i)^{n+1}}$$

$\left|\frac{-(z-i)}{2i}\right| = \frac{|z-i|}{2} < 1$ because $|z-i| < 2$

So, if $z \in \Omega$, then

$$f(z) = \left(\frac{1}{z-i}\right)\left[\frac{1}{2}\cdot\frac{1}{z-i} + \frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n(z-i)^n}{(2i)^{n+1}}\right]$$

$$= \boxed{\frac{1}{2}\cdot\frac{1}{(z-i)^2} + \frac{1}{2}\cdot\sum_{n=0}^{\infty}\frac{(-1)^n(z-i)^{n-1}}{(2i)^{n+1}}}$$

$$= \boxed{\frac{1/2}{(z-i)^2} + \frac{1/4i}{(z-i)}} + \boxed{\frac{1}{8} - \frac{1}{16i}(z-i) + \cdots}$$

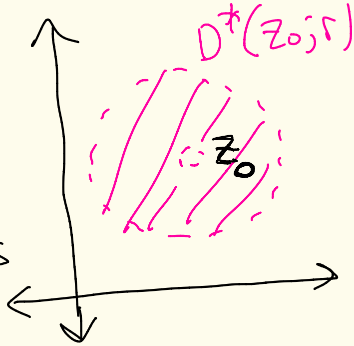
$z_0 = i$ is a pole of order 2.

$$\boxed{\mathrm{Res}(f;i) = \frac{1}{4i}}$$

Theorem (Removable Singularity Thm)

Let $z_0 \in \mathbb{C}$. Suppose that z_0 is an isolated singularity of a function f [that is, f is analytic in an r-neighborhood $D^*(z_0;r)$ of z_0 but not analytic at z_0].

Then, z_0 is a removable singularity of f iff one of the following conditions hold:



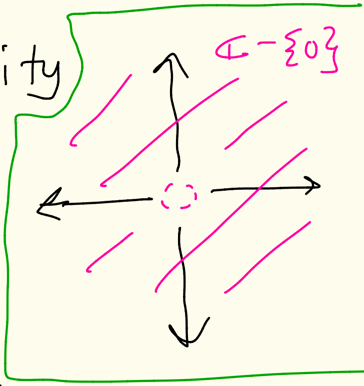
① f is bounded in some deleted ε-neighborhood of z_0

② $\lim_{z \to z_0} f(z)$ exists

③ $\lim_{z \to z_0} (z-z_0) f(z) = 0$

$\boxed{\text{Proof is after example}}$

Ex: Let $f(z) = \dfrac{\sin(z)}{z}$



f has an isolated singularity at $z_0 = 0$.

f is analytic on $\mathbb{C}-\{0\}$.

Using part ③ of the previous theorem we see that

$$\lim_{z \to 0} (z-0)\frac{\sin(z)}{z} = \lim_{z \to 0} \sin(z)$$

$$= \lim_{z \to 0} \frac{e^{iz}-e^{-iz}}{2i} = \frac{e^{i0}-e^{-i0}}{2i} = 0$$

So we have a removable singularity.

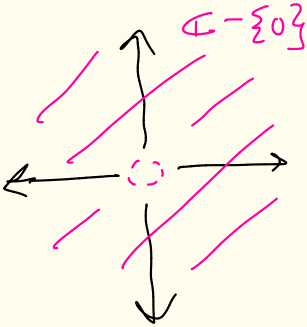
Note that if $z \neq 0$, then

$$f(z) = \frac{\sin(z)}{z} = \frac{1}{z}\left[z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \cdots\right]$$

$$= 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

$$f(z) = \frac{\sin(z)}{z}$$

analytic on $\mathbb{C} - \{0\}$

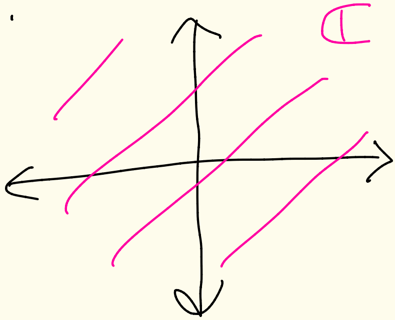


Note: $\tilde{f}(0) = 1$

Let

$$\tilde{f}(z) = 1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \cdots$$

$\tilde{f}(z)$'s power series converges on all of $\mathbb{C}$.



Thus, $\tilde{f}$ is analytic on $\mathbb{C}$.

And $\tilde{f}(z) = f(z) \quad \forall z \in \mathbb{C} - \{0\}$

So, $\tilde{f}$ extends f to all of $\mathbb{C}$ essentially removing the singularity.

proof of ($\Leftarrow$):

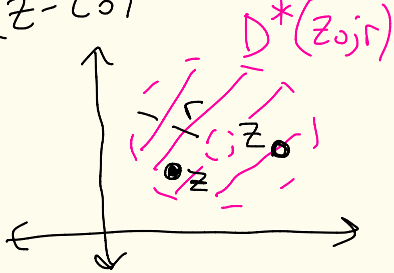
Let z_0 be an isolated singularity of f where f is analytic in some deleted r-neighborhood of z_0.

One can show that either of conditions 1 or 2 imply condition 3.

So we will show condition 3 implies that we have a removable singularity,

We know f has a Laurent series in $D^*(z_0;r)$ and it looks like

$$f(z) = \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n} + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

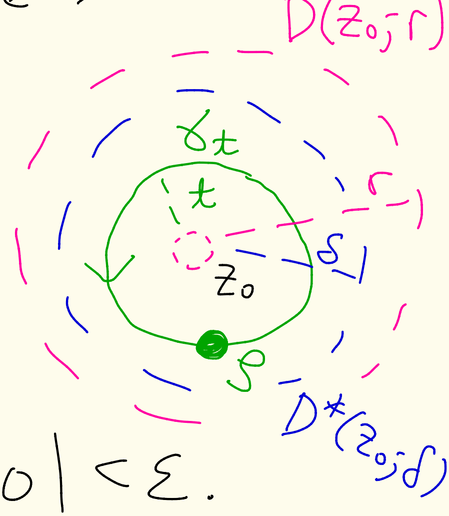


From Laurent series formula

$$b_n = \frac{1}{2\pi i} \int_{\gamma_t} f(\rho)(\rho - z_0)^{n-1}\, d\rho \qquad (n \geq 1)$$

where γ_t is a circle centered at z_0, oriented counterclockwise, with radius t where $0 < t < r$.

Furthermore, let's assume $t < 1$.



Let $\varepsilon > 0$.

Since $\lim_{\rho \to 0} f(\rho)(\rho - z_0) = 0$

there exists $\delta > 0$

where if $\rho \in D^*(z_0; \delta)$

then $|f(\rho)(\rho - z_0) - 0| < \varepsilon$.

Shrink t if needed so that $t < \delta$.

Thus, if ρ is on γ_t then

$$|f(\rho)| < \frac{\varepsilon}{|\rho - z_0|} = \frac{\varepsilon}{t}$$

Thus,

$$|b_n| = \left| \frac{1}{2\pi i} \int_{\gamma_t} f(\rho)(\rho - z_0)^{n-1} d\rho \right|$$

$$= \frac{1}{2\pi} \left| \int_{\gamma_t} f(\rho)(\rho - z_0)^{n-1} d\rho \right|$$

$$\leq \frac{1}{2\pi} \cdot \frac{\varepsilon}{t} \cdot t^{n-1} \cdot \underbrace{2\pi t}_{\text{length of } \gamma_t}$$

ρ is on γ_t

$|f(\rho)| < \frac{\varepsilon}{t}$

$|(\rho - z_0)^{n-1}| = |\rho - z_0|^{n-1} = t^{n-1}$

$$= \varepsilon t^{n-1} < \varepsilon$$

$\boxed{t < 1}$

So, $|b_n| < \varepsilon$ for any $\varepsilon > 0$.
Thus, $b_n = 0$ for all n. So, z_0 is a removable singularity. ∎

Theorem: Let g and h be analytic at z_0 [ie, g and h are analytic in some disc around z_0].

Suppose g has a zero of order $m \geq 0$ at z_0 and h has a zero of order $k > 0$ at z_0.

[If $m = 0$, we mean that $g(z_0) \neq 0$.]

(i) If $m \geq k$, then $f(z) = \dfrac{g(z)}{h(z)}$ has a removable singularity at z_0

(ii) If $m < k$ then $f(z) = \dfrac{g(z)}{h(z)}$ has a pole of order $k - m$ at z_0.

$\boxed{\text{proof is after two examples}}$

Ex: $f(z) = \dfrac{\sin(z)}{z} = \dfrac{g(z)}{h(z)}$

where $g(z) = \sin(z)$ and $h(z) = z$

$$g(z) = z - \frac{z^3}{3!} + \frac{z^5}{5!} - \frac{z^7}{7!} + \dots$$

$$= z \left[1 - \frac{z^2}{3!} + \frac{z^4}{5!} - \frac{z^6}{7!} + \dots \right]$$

$(z-0)^1$

So, g has a zero at $z_0 = 0$ of order $m = 1$

$h(z) = z$ has a zero at $z_0 = 0$ of order $k = 1$

Since $m \geq k$ we have a removable singularity (case (i) from thm) at $z_0 = 0$.

Ex:

$$f(z) = \frac{(e^z - 1)^2}{z} = \frac{g(z)}{h(z)}$$

f has an isolated singularity at $z_0 = 0$

For any $z \in \mathbb{C}$ we have:

$$g(z) = \left[\left(1 + z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right) - 1\right]^2$$

$$= \left[z + \frac{z^2}{2!} + \frac{z^3}{3!} + \dots\right]^2$$

$$= \left(z + \frac{z^2}{2} + \frac{z^3}{6} + \dots\right)\left(z + \frac{z^2}{2} + \frac{z^3}{6} + \dots\right)$$

$$= z^2 + \left(\frac{1}{2} + \frac{1}{2}\right)z^3 + \left(\frac{1}{6} + \frac{1}{4} + \frac{1}{6}\right)z^4 + \dots$$

$$= \underbrace{z^2}_{(z-0)^2} \underbrace{\left[1 + z + \frac{7}{12}z^2 + \dots\right]}_{\varphi(z)} = z^2 \varphi(z)$$

Where $\varphi(z)$ is analytic on all of $\mathbb{C}$ and $\varphi(0)=1\neq 0$ (23)

If $z\neq 0$, then

$$f(z)=\frac{(e^z-1)^2}{z}=\frac{z^2\varphi(z)}{z}$$

zero of order $m=2$

zero of order $k=1$

$$=\underbrace{z\varphi(z)}_{\tilde{f} \text{ from last week}}$$

$z\varphi(z)$ is analytic on all of $\mathbb{C}$ even though f isn't

f has a removable singularity at $z_0=0$.

Ex: $m(z)=\dfrac{z}{(e^z-1)^2}$

zero of order $m=1$

zero of order $k=2$

$$=\frac{1}{z}\cdot\underbrace{\frac{1}{\varphi(z)}}$$

analytic at $z_0=0$ because $\varphi(0)\neq 0$

Pole of order $k-m=1$

We now give the proof of this theorem stated earlier

Theorem: Let g and h be analytic at z_0. Suppose g has a zero of order $m \geq 0$ at z_0 and h has a zero of order $k > 0$ at z_0. [If $m=0$, this means $g(z_0) \neq 0$].

(i) If $m \geq k$, then $f(z) = \dfrac{g(z)}{h(z)}$ has a removable singularity at z_0.

(ii) If $m < k$, then $f(z) = \dfrac{g(z)}{h(z)}$ has a pole of order $k-m$ at z_0.

proof: We know $g(z) = (z-z_0)^m \varphi_1(z)$ and $h(z) = (z-z_0)^k \varphi_2(z)$ where $\varphi_1(z_0) \neq 0$ and $\varphi_2(z_0) \neq 0$ and φ_1 and φ_2 are analytic at z_0.

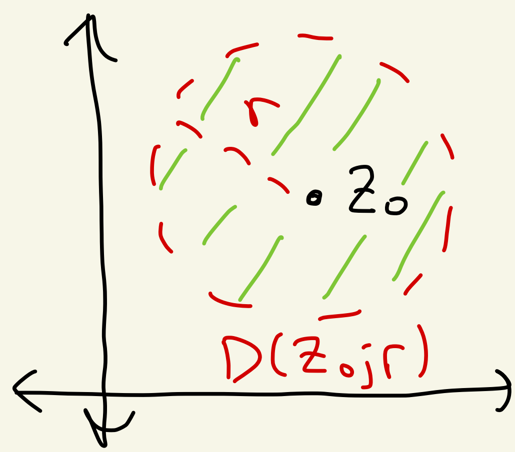
Since φ_1, φ_2 are analytic at z_0 there exists $\hat{r} > 0$ where

φ_1 and φ_2 are analytic on $D(z_0; \hat{r})$.

By Math 4680 since φ_2 is continuous at z_0 and $\varphi_2(z_0) \neq 0$ there exists $\hat{\hat{r}} > 0$ where $\varphi_2(z) \neq 0$ for all $z \in D(z_0; \hat{\hat{r}})$.

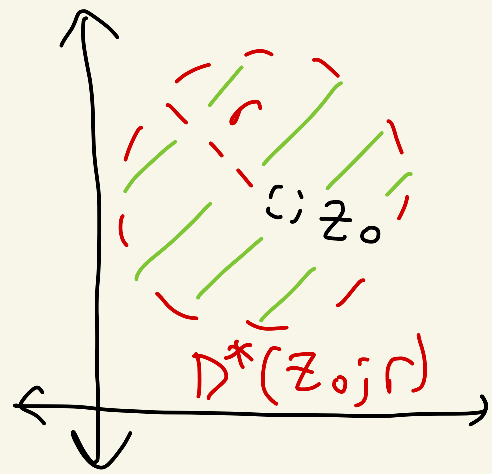
Let $r = \min\{\hat{r}, \hat{\hat{r}}\}$.

So, φ_1, φ_2 are analytic on $D(z_0; r)$ and $\varphi_2(z) \neq 0$ on $D(z_0; r)$.



Thus, if $z \in D^*(z_0; r)$ then

$$f(z) = \frac{g(z)}{h(z)} = \frac{(z-z_0)^m \varphi_1(z)}{(z-z_0)^k \varphi_2(z)}.$$



<u>case (i) - Suppose $m \geq k$.</u>

Then if $z \in D^*(z_0; r)$ then

$$f(z) = (z-z_0)^{m-k}\left[\frac{\varphi_1(z)}{\varphi_2(z)}\right].$$

But also since $m-k \geq 0$ we know

$$(z-z_0)^{m-k}\left[\frac{\varphi_1(z)}{\varphi_2(z)}\right] \text{ is analytic}$$

in <u>all</u> of $D(z_0; r)$.

So, $$(z-z_0)^{m-k}\left[\frac{\varphi_1(z)}{\varphi_2(z)}\right] = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for $z \in D(z_0; r)$

But that means

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

for all $z \in D^*(z_0; r)$.

So, this is f's Laurent series at z_0.

So, we have a removable singularity at z_0.

<u>Case (ii) - Suppose $k > m$</u>

Then if $z \in D^*(z_0; r)$ we have

$$f(z) = \frac{(z - z_0)^m \varphi_1(z)}{(z - z_0)^k \varphi_2(z)}$$

$$= \frac{\left(\varphi_1(z) / \varphi_2(z)\right)}{(z - z_0)^{k-m}}$$

$k > m$
so $k - m > 0$

Since φ_1, φ_2 are analytic in $D(z_0;r)$ and $\varphi_2(z) \neq 0$ when $z \in D(z_0;r)$ we know $\frac{\varphi_1}{\varphi_2}$ is analytic in $D(z_0;r)$.

So, $$\frac{\varphi_1(z)}{\varphi_2(z)} = \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for all $z \in D(z_0;r)$.

Also, $a_0 = \frac{\varphi_1(z_0)}{\varphi_2(z_0)} \neq 0$ because $\varphi_1(z_0) \neq 0$.

Thus, if $z \in D^*(z_0;r)$ then

$$f(z) = \frac{\varphi_1(z)/\varphi_2(z)}{(z-z_0)^{k-m}}$$

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$$= \frac{1}{(z-z_0)^{k-m}} \left[\sum_{n=0}^{\infty} a_n (z-z_0)^n \right]$$

$$= \frac{1}{(z-z_0)^{k-m}} \left[a_0 + a_1 (z-z_0) + \cdots \right]$$

$$= \frac{a_0}{(z-z_0)^{k-m}} + \frac{a_1}{(z-z_0)^{k-m-1}} + \cdots + \frac{a_{k-m-1}}{(z-z_0)}$$

$$+ a_{k-m} + a_{k-m+1}(z-z_0) + \cdots$$

Since $a_0 \neq 0$ we see we have a pole of order $k-m$.

$\blacksquare$

Theorem (On poles of order m)

Let f have an isolated singularity at $z_0 \in \mathbb{C}$ [so, f is analytic in a deleted neighborhood of z_0, but not analytic at z_0].

Then z_0 is a pole of order $m \geq 1$ iff $f(z)$ can be written in the form

$$f(z) = \frac{\varphi(z)}{(z - z_0)^m}$$

in some deleted neighborhood $D^*(z_0; r) = D(z_0; r) - \{z_0\}$ of z_0 where $\varphi(z)$ is analytic in $D(z_0; r)$ and $\varphi(z_0) \neq 0$.

Moreover if this is the case then

$$\text{Res}(f; z_0) = \frac{\varphi^{(m-1)}(z_0)}{(m-1)!}.$$

<u>proof:</u>

($\Leftarrow$) Suppose there exists r, φ where

$$f(z) = \frac{\varphi(z)}{(z-z_0)^m} \quad \text{for all } z \in D^*(z_0; r)$$

and φ is analytic in $D(z_0; r)$

and $\varphi(z_0) \neq 0$.

From the above we can write

$$\varphi(z) = \sum_{n=0}^{\infty} \frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^n$$

for all $z \in D(z_0; r)$

So if $z \in D^*(z_0; r)$ then

$$f(z) = \frac{1}{(z-z_0)^m} \varphi(z)$$

(30)

$$= \frac{1}{(z-z_0)^m}\left[\sum_{n=0}^{\infty} \frac{\varphi^{(n)}(z_0)}{n!}(z-z_0)^n\right]$$

$$= \frac{1}{(z-z_0)^m}\left[\varphi(z_0) + \frac{\varphi^{(1)}(z_0)}{1!}(z-z_0) + \dots\right]$$

$$= \frac{\varphi(z_0)}{(z-z_0)^m} + \frac{\varphi^{(1)}(z_0)/1!}{(z-z_0)^{m-1}} + \frac{\varphi^{(2)}(z_0)/2!}{(z-z_0)^{m-2}}$$

$$+ \dots + \frac{\varphi^{(m-1)}(z_0)/(m-1)!}{(z-z_0)} + \frac{\varphi^{(m)}(z_0)}{m!}$$

$$+ \frac{\varphi^{(m+1)}(z_0)}{(m+1)!}(z-z_0) + \dots$$

not 0

b_1 residue

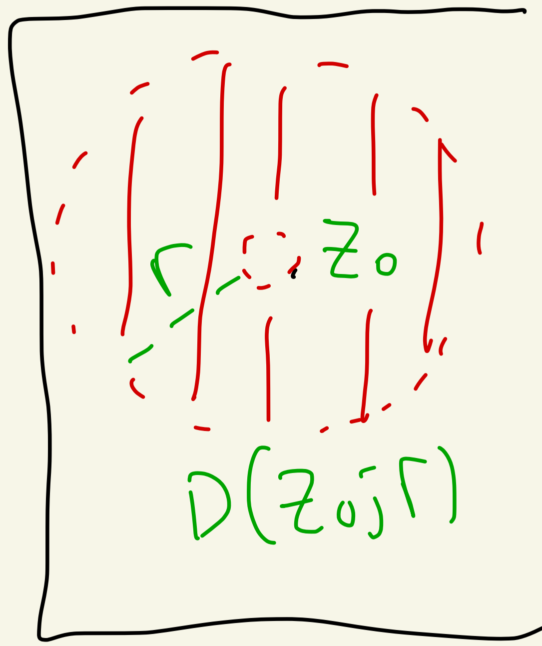
So, f has a pole of order m at z_0 and $\operatorname{Res}(f; z_0) = \dfrac{\varphi^{(m-1)}(z_0)}{(m-1)!}$

($\Rightarrow$) Suppose f has a pole of order m at z_0.

Thus, there exists an $r>0$ where

$$f(z) = \left[\frac{b_m}{(z-z_0)^m} + \cdots + \frac{b_1}{(z-z_0)} \right] + \sum_{n=0}^{\infty} a_n (z-z_0)^n$$

for all $z \in D^*(z_0; r)$ and where $b_m \neq 0$.



Then if $z \in D^*(z_0; r)$ then

$$(z-z_0)^m f(z) = b_m + b_{m-1}(z-z_0) + \cdots + b_1 (z-z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z-z_0)^{n+m}$$

For $z \in D(z_0; r)$ set

$$\varphi(z) = b_m + b_{m-1}(z - z_0) + \dots + b_1 (z - z_0)^{m-1} + \sum_{n=0}^{\infty} a_n (z - z_0)^{n+m}$$

So φ is analytic in $D(z_0; r)$

and $\varphi(z_0) = b_m \neq 0$.

And,

$$f(z) = \frac{\varphi(z)}{(z - z_0)^m}$$

for $z \in D^*(z_0; r)$. $\blacksquare$

Recall: when

f has an isolated singularity at z_0

and $f(z) = \dfrac{\varphi(z)}{(z-z_0)^m}$ where φ is

analytic at z_0 and $\varphi(z_0) \neq 0$

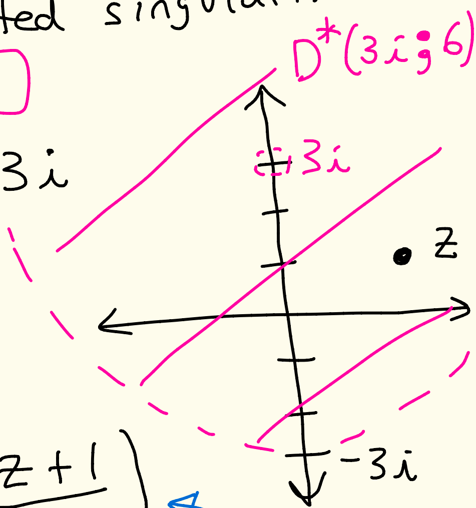
then z_0 is a pole of order m

and $\operatorname{Res}(f; z_0) = \dfrac{\varphi^{(m-1)}(z_0)}{(m-1)!}$.

<u>Ex:</u> Let $f(z) = \dfrac{z+1}{z^2+9}$

$z = \pm 3i$ are the isolated singularities

$[z^2+9=0 \text{ iff } z = \pm 3i]$



Let's look at $z_0 = 3i$

If $z \in D^*(3i, 6)$,

then

$$f(z) = \frac{z+1}{(z-3i)(z+3i)} = \frac{\left(\dfrac{z+1}{z+3i}\right)}{z-3i}$$

$\varphi(z)$

Let $\varphi(z) = \dfrac{z+1}{z+3i}$

Then φ is analytic in $D(3i;6)$

and $\varphi(3i) = \dfrac{3i+1}{3i+3i} = \dfrac{3i+1}{6i} \neq 0$

By the theorem from Monday and today since $f(z) = \dfrac{\varphi(z)}{(z-3i)^1}$

$z_0 = 3i$ is a pole of order 1

and

$$\underbrace{\operatorname{Res}(f;3i)}_{b_1 \text{ term in the Laurent series}} = \frac{\varphi^{(1-1)}(3i)}{(1-1)!} = \varphi(3i)$$

$$= \frac{1+3i}{6i} \cdot \frac{-i}{-i}$$

$\boxed{(-i)(i)=1}$

$$= \frac{-i+3}{6} = \frac{1}{2} - \frac{1}{6}i$$

What about at $z_0 = -3i$

If $z \in D^*(-3i;6)$ then

$$f(z) = \frac{z+1}{(z-3i)(z+3i)} = \frac{\left(\frac{z+1}{z-3i}\right)}{(z-(-3i))^1} \quad \varphi$$

Let $\varphi(z) = \dfrac{z+1}{z-3i}$ which is analytic

at $z_0 = -3i$, and

$$\varphi(-3i) = \frac{-3i+1}{-3i-3i} = \frac{1-3i}{-6i} \neq 0.$$

$z_0 = -3i$ is a pole of order 1

and

$$\text{Res}(f;-3i) = \frac{\varphi^{(1-1)}(-3i)}{(1-1)!} = \frac{\varphi^{(0)}(-3i)}{0!}$$

$$= \varphi(-3i) = \frac{1-3i}{-6i} \cdot \frac{i}{i}$$

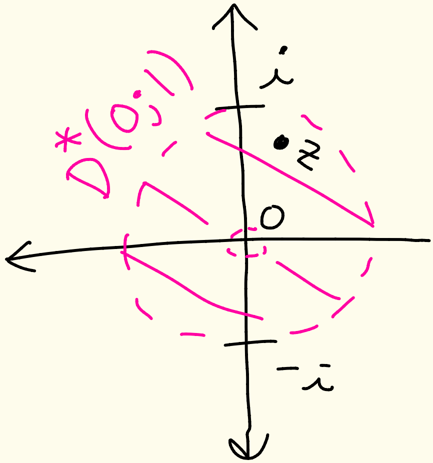
$$= \frac{i+3}{6} = \boxed{\frac{1}{2} + \frac{1}{6}i}$$

Ex: HW 4, part 1

⑥ $f(z) = \dfrac{z+1}{z^3(z^2+1)}$ at $z_0 = 0$

isolated singularities of f are $z_0 = 0, i, -i$

[where $z^3 = 0$ or $z^2+1=0$]



Given $z \in D^*(0;1)$ we have

$$f(z) = \frac{\underbrace{\left(\dfrac{z+1}{z^2+1}\right)}_{\varphi(z)}}{\underbrace{z^3}_{(z-0)^3}}$$

Let $\varphi(z) = \dfrac{z+1}{z^2+1}$. φ is analytic at $z_0 = 0$ and $\varphi(0) = \dfrac{0+1}{0^2+1} = 1 \neq 0$.

So, f has a pole of order 3 at $z_0 = 0$.

So,

$$\mathrm{Res}(f;0) = \frac{\varphi^{(3-1)}(0)}{(3-1)!} = \frac{\varphi^{(2)}(0)}{2!}$$

We have

$$\varphi(z) = \frac{z+1}{z^2+1}$$

$$\varphi'(z) = \frac{(1)(z^2+1) - (z+1)(2z)}{(z^2+1)^2} = \frac{-z^2-2z+1}{(z^2+1)^2}$$

$$\varphi''(z) = \frac{(-2z-2)(z^2+1)^2 - (-z^2-2z+1)\cdot 2(z^2+1)\cdot 2z}{\left((z^2+1)^2\right)^2}$$

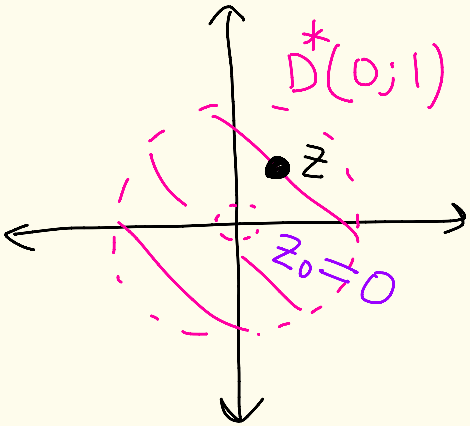
$$\varphi''(0) = \frac{(-2)(1)^2 - (1)\cdot 2(1)\cdot 0}{1^2} = -2$$

Thus, $\mathrm{Res}(f;0) = \dfrac{\varphi''(0)}{2!} = \dfrac{-2}{2} = -1$

Let's find the Laurent series and b_1 directly.

Let $z \in D^*(0;1)$.

So, $0 < |z| < 1$.



Then,

$$f(z) = \frac{1+z}{z^3(z^2+1)}$$

$$= (1+z) \cdot \frac{1}{z^3} \cdot \frac{1}{1+z^2}$$

$$= (1+z) \cdot \frac{1}{z^3} \cdot \frac{1}{1-(-z^2)}$$

$$= (1+z) \cdot \frac{1}{z^3} \cdot \left[1 + (-z^2) + (-z^2)^2 + (-z^2)^3 + \dots \right]$$

need $|-z^2| < 1$
or $|z^2| < 1$
or $|z| < 1$ which is the case

$$= (1+z)\left[\frac{1}{z^3} - \frac{1}{z} + z - z^3 + z^5 - \dots\right]$$

$$= \frac{1}{z^3} - \frac{1}{z} + z - z^3 + z^5 - \dots$$
$$+ \frac{1}{z^2} - 1 + z^2 - z^4 + z^6 - \dots$$

$$= \frac{1}{z^3} + \frac{1}{z^2} - \frac{1}{z} - 1 + z + z^2 - z^3 - z^4 + z^5 + z^6 - \dots$$

pole of order 3

$\mathrm{Res}(f;0) = b_1 = -1$

Theorem (on simple poles)

Suppose that f has an isolated singularity at $z_0 \in \mathbb{C}$.

[So, f is analytic in a deleted neighborhood of z_0 and not analytic at z_0]

Then, z_0 is a simple pole of f iff $\lim_{z \to z_0} (z - z_0) f(z)$

exists and is non-zero.

Moreover, if z_0 is a simple pole then

$$\text{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z)$$

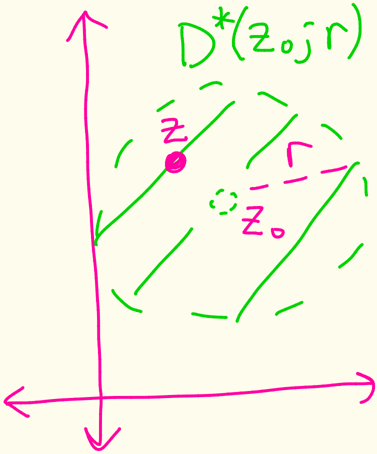
<u>pf:</u> ($\Rightarrow$) Suppose f has a simple pole at z_0.

Then there exists $r>0$ where

$$f(z) = \frac{\overbrace{b_1}^{Res(f;z_0)}}{(z-z_0)} + a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

for all $z \in D^*(z_0;r)$

where $b_1 \neq 0$



Then,

$$\lim_{z\to z_0} (z-z_0)f(z)$$

$$= \lim_{z\to z_0}\left[b_1 + a_0(z-z_0) + a_1(z-z_0)^2 + \cdots \right]$$

$$= b_1 + a_0(0) + a_1(0)^2 + \cdots$$

$$= b_1 \neq 0.$$ And, $\lim_{z\to z_0} (z-z_0)f(z) = Res(f;z_0)$

($\Leftarrow$) Suppose $\lim_{z\to z_0} (z-z_0)f(z)$ exists

and is non-zero.

Let $g(z) = (z-z_0)f(z)$.

Since f has an isolated singularity at z_0, so does g.

Then,

$$\lim_{z\to z_0} (z-z_0)g(z) = \underbrace{\left[\lim_{z\to z_0} (z-z_0)\right]}_{0}\underbrace{\left[\lim_{z\to z_0} g(z)\right]}_{\text{non-zero}}$$

$$= 0$$

By a previous thm, this means that g has a removable singularity at z_0.

Thus, for some $r>0$

$$g(z) = a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots$$

for all $z \in D^*(z_0; r)$.

Then f's Laurent series at z_0 is

$$f(z) = \frac{g(z)}{(z-z_0)}$$

$$= \frac{a_0}{(z-z_0)} + a_1 + a_2(z-z_0) + a_3(z-z_0)^2 + \cdots$$

Also,

$(\Leftarrow)$ assumption

$$0 \neq \lim_{z \to z_0} (z-z_0) f(z)$$

$$= \lim_{z \to z_0} \left[a_0 + a_1(z-z_0) + a_2(z-z_0)^2 + \cdots \right]$$

$$= a_0 + a_1(0) + a_2(0) + \cdots$$

$$= a_0 .$$

So, f has a simple pole at z_0 $\blacksquare$

Ex: $f(z) = \dfrac{\cos(z)}{z}$

At $z_0 = 0$,

$$\lim_{z \to 0} \underbrace{z \cdot \frac{\cos(z)}{z}}_{(z-0)f(z)} = \lim_{z \to 0} \cos(z)$$

$$= \cos(0) = 1 \neq 0$$

Thus, f has a simple pole at $z_0 = 0$ and $\operatorname{Res}(f; 0) = 1$.

Theorem: Let g and h be analytic at z_0 [ie they are analytic in a disc around z_0]

Suppose $g(z_0) \neq 0$, $h(z_0) = 0$, and $h'(z_0) \neq 0$.

Then, $f(z) = \dfrac{g(z)}{h(z)}$ has a simple pole at z_0 and

$$\operatorname{Res}(f; z_0) = \frac{g(z_0)}{h'(z_0)}$$

proof: Since $h(z_0) = 0$ we have

$$\lim_{z \to z_0} \frac{h(z)}{z - z_0} = \lim_{z \to z_0} \left[\frac{h(z) - h(z_0)}{z - z_0} \right]$$

$$= h'(z_0) \neq 0$$

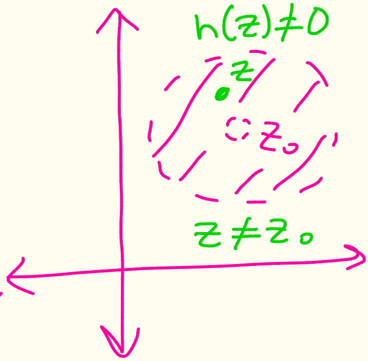
$\boxed{h(z_0) = 0}$

Thus,

$$\lim_{z \to z_0} \frac{z - z_0}{h(z)} = \frac{1}{h'(z_0)}$$

This makes sense because $h'(z_0) \neq 0$ and $h(z_0) = 0$ so h's power series at z_0 will look like $h(z) = 0 + h'(z_0)(z - z_0) + \cdots$

Thus, h isn't identically 0 around z_0 and so as we discussed before z_0 will be an isolated zero of h.



Thus,

$$\lim_{z \to z_0} (z - z_0) f(z)$$

$$= \lim_{z \to z_0} (z - z_0) \cdot \frac{g(z)}{h(z)}$$

$$= \lim_{z \to z_0} \frac{(z - z_0)}{h(z)} \cdot g(z) = \frac{g(z_0)}{h'(z_0)} \neq 0$$

$g(z_0) \neq 0$

Thus, by the previous theorem f has a simple pole at z_0 and

$$\operatorname{Res}(f; z_0) = \lim_{z \to z_0} (z - z_0) f(z) = \frac{g(z_0)}{h'(z_0)}$$

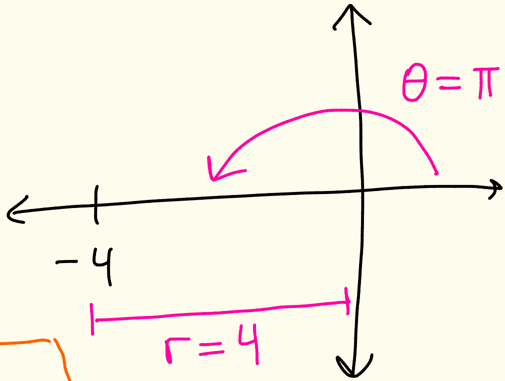
$\blacksquare$

Ex: Let $f(z) = \dfrac{z}{z^4+4}$

Let's find the singularities of f.

$$z^4 + 4 = 0$$

$$z^4 = -4 = \underbrace{4e^{\pi i}}_{re^{i\theta}}$$



solutions!

$$\boxed{z_k = 4^{1/4} e^{\left(\frac{\pi}{4} + \frac{2\pi}{4}k\right)i}, \quad k = 0, 1, 2, 3}$$

$$z_0 = \sqrt{2}\, e^{\frac{\pi}{4} i} = \sqrt{2}\left(\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = 1 + i$$

$$z_1 = \sqrt{2}\, e^{\frac{3\pi}{4} i} = \sqrt{2}\left(-\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2}\right) = -1 + i$$

$$z^n = w$$
$$w = re^{i\theta}$$
$$z_k = r^{1/n} e^{\left(\frac{\theta}{n} + \frac{2\pi k}{n}\right)i}$$
$$k = 0, 1, 2, \ldots, n-1$$

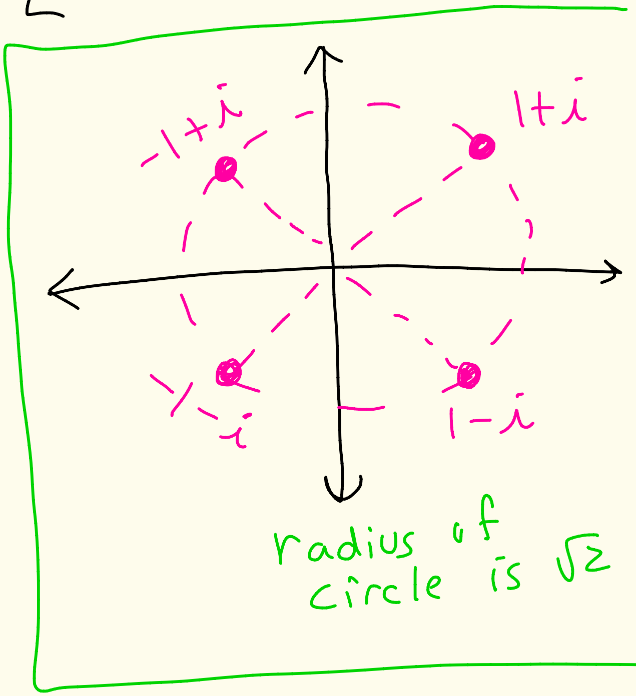
$$\boxed{e^{i\theta} = \cos(\theta) + i\sin(\theta)}$$

$$z_2 = \sqrt{2}\, e^{\frac{5\pi}{4} i} = \sqrt{2}\left[-\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right]$$

$$= -1 - i$$

$$z_3 = \sqrt{2}\, e^{\frac{7\pi}{4} i} = \sqrt{2}\left[\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}\right] = 1 - i$$

The isolated singularities of f are $1+i, -1+i, -1-i, 1-i$.



They all turn out to be simple poles. Let's check this for one of them.

$$f(z) = \frac{z}{z^4+4} = \frac{g(z)}{h(z)}$$

where $g(z) = z$ and $h(z) = z^4+4$

g and h are both analytic at $z_0 = 1+i$

And,

$g(1+i) = 1+i \neq 0$

$h(1+i) = (1+i)^4 + 4 = 0$ from previous page

$h'(1+i) = 4(1+i)^3 \neq 0$

Thus, $z_0 = 1+i$ is a simple pole of f and

$$\operatorname{Res}(f; 1+i) = \frac{g(1+i)}{h'(1+i)} = \frac{1+i}{4(1+i)^3}$$

$$= \frac{1}{4(1+i)^2}.$$

You could have also done this:

$$f(z) = \frac{z}{z^4+4}$$

$$= \frac{z}{(z-(1+i))(z-(-1+i))(z-(-1-i))(z-(1-i))}$$

$$= \frac{\left(\frac{z}{(z-(-1+i))(z-(-1-i))(z-(-1-i))}\right)}{z-(1+i)}$$

$\varphi(z)$

φ is analytic at $1+i$ and $\varphi(1+i) \neq 0$.

Then f has a simple pole at $z_0 = 1+i$ and

$$\text{Res}(f; 1+i) = \frac{\varphi^{(1-1)}(1+i)}{(1-1)!} = \varphi(1+i)$$

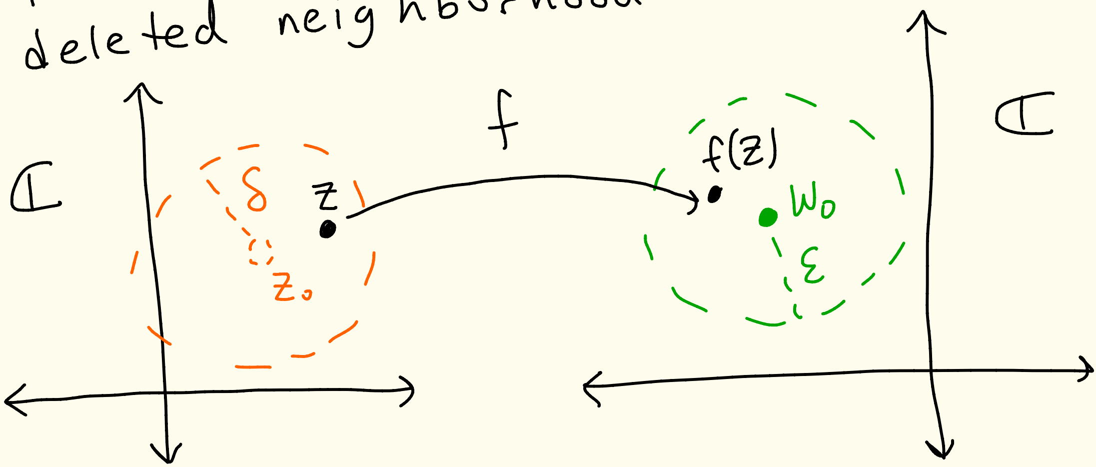
$=$ should be the same

Here are two theorems about essential singularities. If we have time at the end of the course we can prove the first one.

<u>Theorem:</u> (Casorati – Weierstrass)

Suppose z_0 is an essential singularity of a function f. Let w_0 be any complex number. Let $\varepsilon > 0$.

Then the inequality $|f(z) - w_0| < \varepsilon$ is satisfied at some z in each deleted neighborhood $D^*(z_0; \delta)$.



Theorem: (Picard's Theorem)

Let f have an essential singularity at z_0 and let $D^*(z_0;\delta)$ be any deleted neighborhood of z_0.
Then for all $w \in \mathbb{C}$, except perhaps one value, the equation $f(z)=w$ has infinitely many solutions in $D^*(z_0;\delta)$.