Topic 1 – Vector Spaces

Def: A *field* F is a set with two binary operations denoted by $+$ and $\cdot$, such that the following are true.

(F1) For every $a, b \in F$, there exist unique elements $a+b$ and $a \cdot b$ in F.

(F2) For every $a, b, c \in F$ we have

$a+b=b+a$ $a \cdot b = b \cdot a$ (commutative properties)	$a+(b+c)=(a+b)+c$ $a\cdot(b\cdot c)=(a\cdot b)\cdot c$ (associative properties)	$a\cdot(b+c)$ $= a\cdot b + a\cdot c$ $(b+c)\cdot a =$ $b\cdot a + c\cdot a$ (distributive properties)

(F3) There exists elements 0 and 1 in F where $a+0=0+a=a$ and $a\cdot 1 = 1\cdot a = a$ for all a in F.

(F4) For every $a \in F$ there exists $d \in F$ where $a+d=d+a=0$.

(F5) For every $a \in F$ with $a \neq 0$, there exists $f \in F$ where
$$a \cdot f = f \cdot a = 1$$

HW: 0, 1, d, f from F3/F4/F5 are unique.

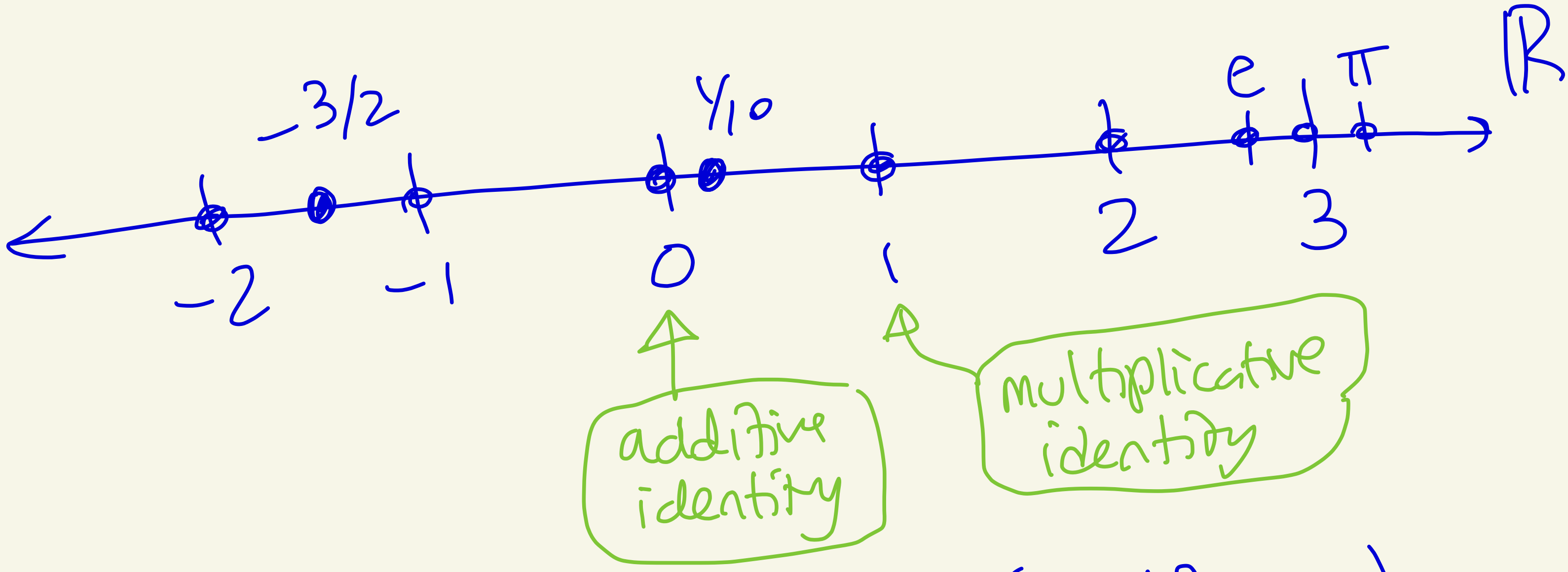
We call 0 the additive identity of F.

We call 1 the multiplicative identity of F.

We denote d in F4 as $-a$ and call it the additive inverse of a.

We denote f in F5 as a^{-1} and call it the multiplicative inverse of a.

Ex: $F = \mathbb{R}$ the set of real numbers is a field.



$$a = \frac{1}{2}, \quad -a = -\frac{1}{2}$$ (additive inverse)

$$a = \pi, \quad a^{-1} = \frac{1}{\pi}$$ (multiplicative inverse)

Ex: $F = \mathbb{Q} = \left\{ \frac{a}{b} \;\middle|\; a, b \in \mathbb{Z},\ b \neq 0 \right\}$

$$= \left\{ -1, 0, 1, 10, \frac{1}{2}, \frac{-3}{7}, \ldots \right\}$$

[rational numbers]

is a field.

$$a = \frac{-3}{7}, \quad -a = \frac{3}{7}, \quad a^{-1} = \frac{-7}{3}$$

Ex:

$$F = \mathbb{C} = \{ x+iy \mid x,y \in \mathbb{R} \}$$
$$= \{ 1 = 1+0i,\ 0 = 0+0i,\ \tfrac{1}{2} = \tfrac{1}{2}+0i,\ 1+i, \ldots \}$$

[$i^2 = -1$]

$\mathbb{C}$ is a field.

[3450, 4550, 4460]

Ex: If p is a prime, then

$$\mathbb{Z}_p = \{ \bar{0}, \bar{1}, \bar{2}, \ldots, \overline{p-1} \}$$

is a field. [$\mathbb{Z}_p$ is called the integers modulo p.]

[We won't use $\mathbb{Z}_p$ in this class]

Def: Let F be a field.
A vector space over F is a set V with two operations. The first operation is addition which takes two elements $v_1, v_2 \in V$ and produces a unique element $v_1 + v_2 \in V$.
The second operation is called scalar multiplication, which takes one element $a \in F$ and one element $v \in V$ and produces a unique element $av \in V$.

could write $a \cdot v$

[The set V is sometimes called the set of "vectors" and F is sometimes called the "scalars"]

The following properties must hold:

(V1) For all $v_1, v_2 \in V$ we have $v_1 + v_2 = v_2 + v_1$.

[commutative property]

(V2) For every $v_1, v_2, v_3 \in V$ we have $v_1 + (v_2 + v_3) = (v_1 + v_2) + v_3$

[associative property]

(V3) There exists an element $\vec{0}$ in V where $\vec{0} + w = w + \vec{0} = w$ for all $w \in V$.

(V4) For every $w \in V$ there exists $z \in V$ with

$$w + z = z + w = \vec{0}$$

(V5) For each $w \in V$ we have

$$1w = w$$

[Here 1 is from F]

(V6) For every $a, b \in F$ and $w \in V$ we have

$$(ab)w = a(bw)$$

V7 For all $a \in F$ and $v_1, v_2 \in V$ we have

$$a(v_1 + v_2) = av_1 + av_2$$

V8 For all $a, b \in F$ and $w \in V$ we have $(a+b)w = aw + bw$

<u>Note:</u> Later we will show that $\vec{0}$ from V3 and the z from V4 are unique.

$\vec{0}$ is called the <u>zero vector</u> in V

z is called the <u>additive inverse of w</u> and will be written $z = -w$.

Ex: $F = \mathbb{R}$,

$$V = \mathbb{R}^2 = \{ (x,y) \mid x, y \in \mathbb{R} \}$$

Then $V = \mathbb{R}^2$ is a vector space over $F = \mathbb{R}$.

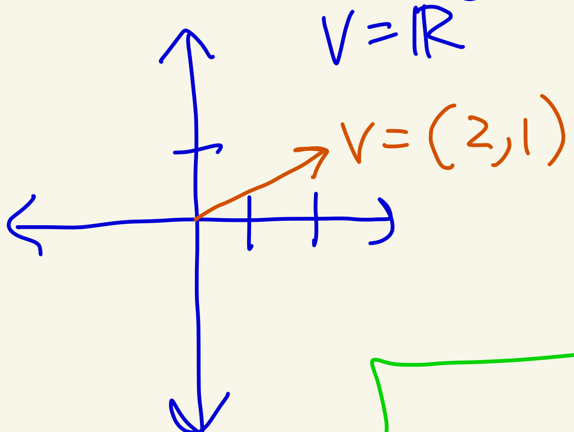
Where

$$(a,b) + (x,y) = (a+x, b+y) \quad \leftarrow \boxed{\text{vector addition}}$$

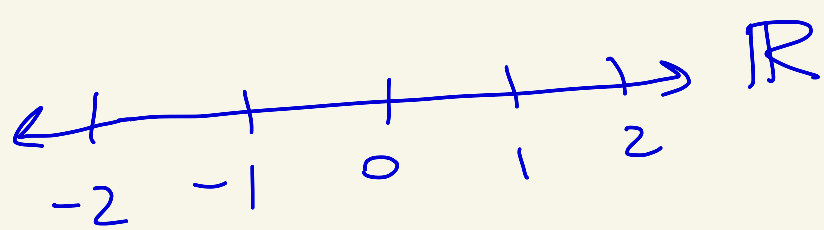
$$\alpha(x,y) = (\alpha x, \alpha y) \quad \leftarrow \boxed{\text{scalar multiplication}}$$

$\boxed{\alpha = \text{alpha}}$

Vectors



scalars/field



Example: $(5,-1) + (2,7) = (7,6)$

$$3(5,-1) = (15,-3)$$

Ex: Let F be a field.

Let

$$V = F^n = \{(x_1, x_2, \ldots, x_n) \mid x_1, x_2, \ldots, x_n \in F\}$$

where $n \geq 1$.

Then $V = F^n$ is a vector space over F using the following operations.

Let $\alpha \in F$ and

$$v = (a_1, a_2, \ldots, a_n)$$
$$w = (b_1, b_2, \ldots, b_n)$$

define vector addition as

$$v + w = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n)$$

and scalar multiplication as

$$\alpha v = (\alpha a_1, \alpha a_2, \ldots, \alpha a_n)$$

proof: Let $\alpha, \beta \in F$ and $v, w, z \in V = F^n$ where $v = (v_1, v_2, \ldots, v_n)$, $w = (w_1, w_2, \ldots, w_n)$ and $z = (z_1, z_2, \ldots, z_n)$.

V1 We have that

$$
\begin{aligned}
v + w &= (v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n) \\
&= (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n) \\
&= (w_1 + v_1, w_2 + v_2, \ldots, w_n + v_n) \\
&= (w_1, w_2, \ldots, w_n) + (v_1, v_2, \ldots, v_n) \\
&= w + v
\end{aligned}
$$

Since F is a field $a + b = b + a$ $\forall a, b \in F$ F2 prop

V2 We have that

$$v+(w+z) = (v_1, v_2, \dots, v_n) + \left[(w_1, w_2, \dots, w_n) + (z_1, z_2, \dots, z_n)\right]$$

$$= (v_1, v_2, \dots, v_n) + (w_1+z_1, w_2+z_2, \dots, w_n+z_n)$$

$$= \big(v_1+(w_1+z_1), v_2+(w_2+z_2), \dots, v_n+(w_n+z_n)\big)$$

$$= \big((v_1+w_1)+z_1, (v_2+w_2)+z_2, \dots, (v_n+w_n)+z_n\big)$$

$$= (v_1+w_1, v_2+w_2, \dots, v_n+w_n) + (z_1, z_2, \dots, z_n)$$

$$= \left[(v_1, v_2, \dots, v_n) + (w_1, w_2, \dots, w_n)\right] + (z_1, z_2, \dots, z_n)$$

$$= [v+w]+z$$

F2 prop
$a+(b+c)=(a+b)+c$
$\forall\, a,b,c \in F$

V3 Define

$$\vec{0} = (0, 0, \ldots, 0)$$

where 0 is the zero element of F.

Then,

$$\begin{aligned} z + \vec{0} &= (z_1, z_2, \ldots, z_n) + (0, 0, \ldots, 0) \\ &= (z_1 + 0, z_2 + 0, \ldots, z_n + 0) \\ &= (z_1, z_2, \ldots, z_n) \\ &= z \end{aligned}$$

F3 prop
$a + 0 = 0 + a = a$
$\forall a \in F$

and

$$\begin{aligned} \vec{0} + z &= (0, 0, \ldots, 0) + (z_1, z_2, \ldots, z_n) \\ &= (0 + z_1, 0 + z_2, \ldots, 0 + z_n) \\ &= (z_1, z_2, \ldots, z_n) \\ &= z \end{aligned}$$

V4 Given $v = (v_1, v_2, \dots, v_n)$

consider $-v = (-v_1, -v_2, \dots, -v_n)$

where $-v_i$ is the additive inverse of v_i in F.

$\boxed{\text{Using F4}}$

Then,

$$v + (-v) = (v_1 - v_1, v_2 - v_2, v_n - v_n)$$
$$= (0, 0, \dots, 0) = \vec{0}$$

and

$$(-v) + v = (-v_1 + v_1, -v_2 + v_2, \dots, -v_n + v_n)$$
$$= (0, 0, \dots, 0) = \vec{0}$$

V5 Let 1 be the multiplicative identity of F.

Then,

$$1 \cdot v = 1 \cdot (v_1, v_2, \dots, v_n)$$
$$= (1v_1, 1v_2, \dots, 1v_n)$$
$$\overset{F3}{=} (v_1, v_2, \dots, v_n) = v$$

V6 We have that

$$(\alpha\beta)w = (\alpha\beta)(w_1, w_2, \dots, w_n)$$
$$= ((\alpha\beta)w_1, (\alpha\beta)w_2, \dots, (\alpha\beta)w_n)$$
$$\overset{F2}{=} (\alpha(\beta w_1), \alpha(\beta w_2), \dots, \alpha(\beta w_n))$$
$$= \alpha(\beta w_1, \beta w_2, \dots, \beta w_n)$$
$$= \alpha[\beta(w_1, w_2, \dots, w_n)]$$
$$= \alpha[\beta w]$$

F2: $a(bc) = (ab)c$ $\forall a,b,c \in F$

V7 We have that

$$\alpha(v+w)$$
$$= \alpha\left[(v_1, v_2, \ldots, v_n) + (w_1, w_2, \ldots, w_n)\right]$$
$$= \alpha(v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$
$$= (\alpha(v_1 + w_1), \alpha(v_2 + w_2), \ldots, \alpha(v_n + w_n))$$
$$= (\alpha v_1 + \alpha w_1, \alpha v_2 + \alpha w_2, \ldots, \alpha v_n + \alpha w_n)$$
$$= (\alpha v_1, \alpha v_2, \ldots, \alpha v_n) + (\alpha w_1, \alpha w_2, \ldots, \alpha w_n)$$
$$= \alpha(v_1, v_2, \ldots, v_n) + \alpha(w_1, w_2, \ldots, w_n)$$
$$= \alpha v + \alpha w$$

F2
$a(b+c) = ab + ac$
$\forall a, b, c \in F$

V8 We have that

$$(\alpha+\beta)w = (\alpha+\beta)(w_1, w_2, \ldots, w_n)$$

$$= ((\alpha+\beta)w_1, (\alpha+\beta)w_2, \ldots, (\alpha+\beta)w_n)$$

$$= (\alpha w_1 + \beta w_1, \alpha w_2 + \beta w_2, \ldots, \alpha w_n + \beta w_n)$$

F2
$(a+b)c =$
$ac+bc$
$\forall a,b,c \in F$

$$= (\alpha w_1, \alpha w_2, \ldots, \alpha w_n) + (\beta w_1, \beta w_2, \ldots, \beta w_n)$$

$$= \alpha(w_1, w_2, \ldots, w_n) + \beta(w_1, w_2, \ldots, w_n)$$

$$= \alpha w + \beta w$$

Since V1 – V8 are true, $V = F^n$ is a vector space over F. $\blacksquare$

Ex:

$V = \mathbb{R}^5$ is a vector space over $F = \mathbb{R}$

$V = \mathbb{Q}^{10,000,000}$ is a vector space over $F = \mathbb{Q}$

Ex: Let F be a field.
Let $V = M_{m,n}(F)$ be the set of all $m \times n$ matrices with entries from F. Then one can show that V is a vector space over F where vector addition is defined as

$$\begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} + \begin{pmatrix} b_{11} & b_{12} & \cdots & b_{1n} \\ b_{21} & b_{22} & \cdots & b_{2n} \\ \vdots & \vdots & & \vdots \\ b_{m1} & b_{m2} & \cdots & b_{mn} \end{pmatrix}$$

$$= \begin{pmatrix} a_{11}+b_{11} & a_{12}+b_{12} & \cdots & a_{1n}+b_{1n} \\ a_{21}+b_{21} & a_{22}+b_{22} & \cdots & a_{2n}+b_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1}+b_{m1} & a_{m2}+b_{m2} & \cdots & a_{mn}+b_{mn} \end{pmatrix}$$

[more on next page]

and scalar multiplication is defined as

$$\alpha \begin{pmatrix} a_{11} & a_{12} & \cdots & a_{1n} \\ a_{21} & a_{22} & \cdots & a_{2n} \\ \vdots & \vdots & & \vdots \\ a_{m1} & a_{m2} & \cdots & a_{mn} \end{pmatrix} = \begin{pmatrix} \alpha a_{11} & \alpha a_{12} & \cdots & \alpha a_{1n} \\ \alpha a_{21} & \alpha a_{22} & \cdots & \alpha a_{2n} \\ \vdots & \vdots & & \vdots \\ \alpha a_{m1} & \alpha a_{m2} & \cdots & \alpha a_{mn} \end{pmatrix}$$

Where

$$\vec{0} = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}$$

<u>proof:</u> Similar to last example. $\boxtimes$

Ex: $F = \mathbb{R}$

$$V = M_{2,3}(\mathbb{R}) = \left\{ \begin{pmatrix} a & b & c \\ d & e & f \end{pmatrix} \;\middle|\; \begin{matrix} a,b,c,d, \\ e,f \in \mathbb{R} \end{matrix} \right\}$$

$$\vec{0} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

Example of computation is

$$\begin{pmatrix} 1 & 0 & -1 \\ 3 & 2 & 1 \end{pmatrix} + \begin{pmatrix} 5 & 3 & 1 \\ 2 & 0 & \pi \end{pmatrix}$$

$$= \begin{pmatrix} 6 & 3 & 0 \\ 5 & 2 & 1+\pi \end{pmatrix}$$

and

$$\frac{1}{2}\begin{pmatrix} 3 & 0 & 1 \\ \pi & \sqrt{2} & 5 \end{pmatrix} = \begin{pmatrix} \frac{3}{2} & 0 & 1/2 \\ \frac{\pi}{2} & \frac{\sqrt{2}}{2} & \frac{5}{2} \end{pmatrix}$$

Ex: Let $F = \mathbb{R}$ or $F = \mathbb{C}$.
Let $n \geq 0$ be an integer.
Define

$$P_n(F) = \left\{ a_0 + a_1 x + a_2 x^2 + \ldots + a_n x^n \;\middle|\; a_i \in F \right\}$$

So, $P_n(F)$ are all polynomials of degree $\leq n$ with coefficients from the field F.

One can show that $V = P_n(F)$ is a vector space over F where vector addition is given by:

$$(a_0 + a_1 x + \ldots + a_n x^n) + (b_0 + b_1 x + \ldots + b_n x^n)$$
$$= (a_0 + b_0) + (a_1 + b_1) x + \ldots + (a_n + b_n) x^n$$

and scalar multiplication is given by (22)

$$\alpha(a_0 + a_1 x + \dots + a_n x^n)$$
$$= (\alpha a_0) + (\alpha a_1)x + \dots + (\alpha a_n)x^n$$

Note: In $P_n(F)$, the zero vector is $\vec{0} = 0 + 0x + \dots + 0x^n$.

Equality:

We define equality as follows:

Let $f = a_0 + a_1 x + \dots + a_n x^n$

and $g = b_0 + b_1 x + \dots + b_n x^n$.

We define $f = g$ iff

$$a_0 = b_0,\ a_1 = b_1,\ \dots,\ a_n = b_n$$

Ex: Let $F = \mathbb{R}$.

Consider

$$V = P_4(\mathbb{R})$$

$$= \{ a_0 + a_1 x + a_2 x^2 + a_3 x^3 + a_4 x^4 \mid a_i \in \mathbb{R} \}$$

$$= \{ 0, 5, \pi + 3x^2 - x^4, x^4, \cdots \}$$

$5 = 5 + 0x + 0x^2 + 0x^3 + 0x^4$

$0 = 0 + 0x + 0x^2 + 0x^3 + 0x^4$

example of adding:

$$(\pi + 3x^2 - x^4) + (1 - x^2 + x^3)$$

$$= (\pi + 1) + 2x^2 + x^3 - x^4$$

example of scaling:

$$\frac{1}{2}(1 - 6x^2 + x^4) = \frac{1}{2} - 3x^2 + \frac{1}{2}x^4$$

$P_4(\mathbb{R})$ is like $\mathbb{R}^5$

$$1 + x - x^2 + 5x^3 - 7x^4 \longleftarrow \boxed{\text{in } P_4(\mathbb{R})}$$

$$\updownarrow$$

$$(1, 1, -1, 5, -7) \longleftarrow \boxed{\text{in } \mathbb{R}^5}$$

Theorem: Let V be a vector space over a field F.

1. The element $\vec{0}$ from V3 is unique.

[That is, there is only one vector $\vec{0}$ in V that satisfies $\vec{0}+w = w+\vec{0} = w$ for all $w \in V$.]

2. Given $w \in V$, the element z from V4 where $w+z = z+w = \vec{0}$ is unique.

[Recall we write z as $-w$]

proof:

1. Suppose $\vec{0}_1, \vec{0}_2 \in V$ where

$$\vec{0}_1 + w = w + \vec{0}_1 = w$$

and $$\vec{0}_2 + w = w + \vec{0}_2 = w$$

for all $w \in V$.

So, $\vec{0}_1$ and $\vec{0}_2$ are both zero vectors.

Then,

$$\vec{O}_1 \underset{\boxed{w = w + \vec{O}_2}}{=} \vec{O}_1 + \vec{O}_2 \underset{\boxed{\vec{O}_1 + w = w}}{=} \vec{O}_2$$

Thus, $\vec{O}_1 = \vec{O}_2$.

So there can be only one zero vector.

(2) Let $w \in V$.

Suppose $z_1, z_2 \in V$ where

$$w + z_1 = z_1 + w = \vec{O}$$

and $w + z_2 = z_2 + w = \vec{O}$.

So, z_1, z_2 are both additive inverses for w

We have $w + z_1 = \vec{O}$.

Add z_2 to both sides to get

$$z_2 + (w + z_1) = z_2 + \vec{O}$$

Thus, using associativity we have

$$\underbrace{(z_2 + w)}_{\vec{0}} + z_1 = z_2$$

Thus, $\vec{0} + z_1 = z_2$.

So, $z_1 = z_2$.

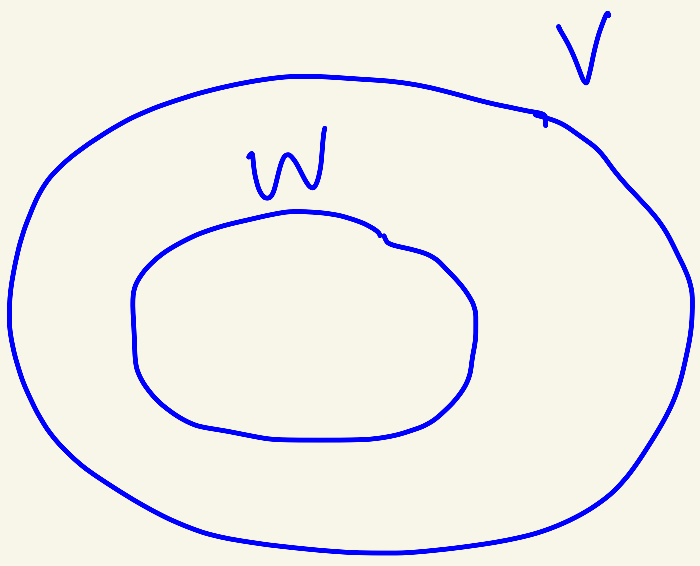
Ergo, there is only one additive inverse for w.

$\blacksquare$

Def: Let V be a vector space over a field F.

Let $W \subseteq V$.

We say that W is a **subspace** of V if W is a vector space over F using the same vector addition and scalar multiplication as in V

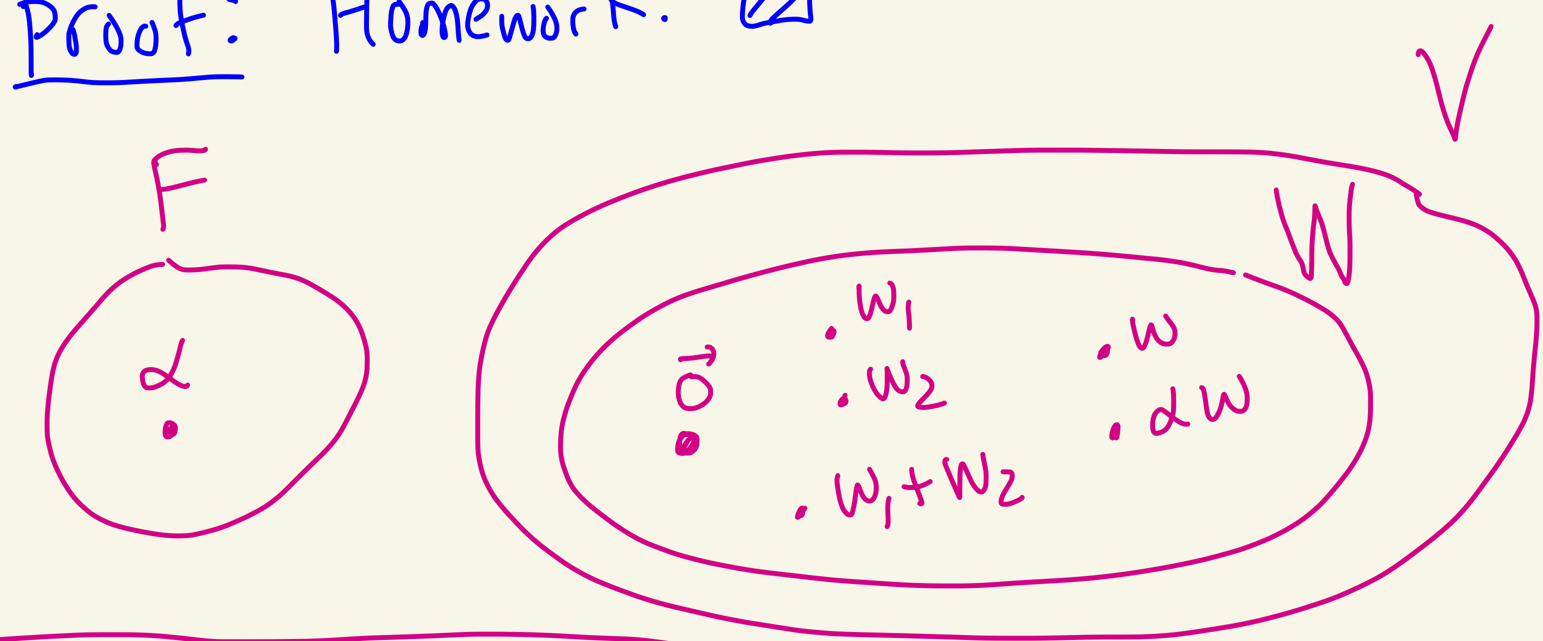


Theorem: Let V be a vector space over a field F. Let W be a subset of V.

W is a subspace of V if and only if the following three conditions hold:

1. $\vec{0} \in W$ — you can actually just show $W \neq \emptyset$
2. If $w_1, w_2 \in W$, then $w_1 + w_2 \in W$. — W is closed under $+$
3. If $\alpha \in F$ and $w \in W$, then $\alpha w \in W$ — W is closed under scaling

Proof: Homework. $\boxed{}$



PICTURE OF ①, ②, ③

Ex: Let $V=\mathbb{R}^3$, $F=\mathbb{R}$.

Let
$$W=\{(0,b,c) \mid b,c\in\mathbb{R}\}$$
$$=\{(0,1,\pi),(0,-1,\sqrt{2}),\ldots\}$$

Is W a subspace of V?

It is!

Let's prove it.

① Setting $b=0$, $c=0$ gives
$(0,b,c)=(0,0,0)$ is in W.
So, $\vec{0}\in W$.

② Let $w_1, w_2 \in W$.
Then, $w_1=(0,b_1,c_1)$ and
$w_2=(0,b_2,c_2)$ where b_1, c_1, b_2, c_2 are in $\mathbb{R}$.

Then,

$w_1 + w_2 = (0, b_1 + b_2, c_1 + c_2)$

which is in W, since $b_1 + b_2, c_1 + c_2 \in \mathbb{R}$

3 Let $\alpha \in \mathbb{R}$ and $w \in W$.

Then, $w = (0, b, c)$ where $b, c \in \mathbb{R}$.

And $\alpha w = (0, \alpha b, \alpha c)$ which is still in W, since $\alpha b, \alpha c \in \mathbb{R}$.

By 1, 2, and 3

W is a subspace of $V = \mathbb{R}^3$.

$\blacksquare$

Ex: Let

$V = P_2(\mathbb{R})$ and $F = \mathbb{R}$.

Let

$$W = \{1 + bx \mid b \in \mathbb{R}\}$$
$$= \{1+2x,\ 1-3x,\ \ldots\}$$

Is W a subspace of $P_2(\mathbb{R})$?

No. For example

$1+2x,\ 1-3x \in W$

but

$$(1+2x)+(1-3x) = \underbrace{2}_{\text{not } 1} - x \notin W$$

Also: • $\vec{0} = 0 + 0x \notin W$

• $1+x \in W$, but $5\cdot(1+x) = 5+5x \notin W$

Note: Let V be a vector space over F.

V has at least these subspaces:

$$W = \{\vec{0}\}$$

trivial subspace of V

$$W = V$$