Topic 1 - Series

$\boxed{\text{Series}}$ (HW 1 topic)

Def: Let n_0 be an integer. Consider the infinite series

$$\sum_{n=n_0}^{\infty} a_n = a_{n_0} + a_{n_0+1} + a_{n_0+2} + \cdots$$

where each a_n is a complex number. From this series we create a sequence $(S_k)_{k=1}^{\infty}$ of partial sums defined by

$$S_k = \sum_{n=n_0}^{n_0+k-1} a_n = a_{n_0} + a_{n_0+1} + \cdots + a_{n_0+k-1}$$

So,

$$S_1 = a_{n_0}$$

$$S_2 = a_{n_0} + a_{n_0+1}$$

$$S_3 = a_{n_0} + a_{n_0+1} + a_{n_0+2}$$

and so on.

We say that $\sum_{n=n_0}^{\infty} a_n$ converges to the limit S, and we write $\sum_{n=n_0}^{\infty} a_n = S$, if $(s_k)_{k=1}^{\infty}$ converges and

$$\lim_{k \to \infty} s_k = S.$$

Otherwise, if $(s_k)_{k=1}^{\infty}$ diverges then we say $\sum_{n=n_0}^{\infty} a_n$ diverges

Ex: Consider the series

$$\sum_{n=0}^{\infty}\left(\frac{i}{3}\right)^n = 1+\frac{i}{3}+\frac{i^2}{3^2}+\frac{i^3}{3^3}+\cdots$$

Does this converge?

Let's look at the partial sums:

$$S_1 = 1$$

$$-\frac{1}{9} \qquad -\frac{i}{27}$$

$$S_2 = 1+\frac{i}{3}$$

$$S_3 = 1+\frac{i}{3}+\frac{i^2}{3^2} = \frac{8}{9}+\frac{i}{3}$$

$$S_4 = 1+\frac{i}{3}+\frac{i^2}{3^2}+\frac{i^3}{3^3} = \frac{8}{9}+\frac{8}{27}i$$

$$\vdots \qquad \vdots$$

Algebra: If $z \neq 1$, then

$$\boxed{1+z+z^2+\cdots+z^k = \frac{1-z^{k+1}}{1-z}}$$

Geometric sum

Proof: cross-multiply

So,

$$S_k = 1 + \frac{i}{3} + \left(\frac{i}{3}\right)^2 + \dots + \left(\frac{i}{3}\right)^{k-1}$$

$$= \frac{1-\left(\frac{i}{3}\right)^k}{1-\frac{i}{3}}$$

$\boxed{z = \frac{i}{3}}$

Later: If $\lim_{n\to\infty}|b_n| = 0$, then $\lim_{n\to\infty} b_n = 0$

Note that $\displaystyle\lim_{k\to\infty}\left|\left(\frac{i}{3}\right)^k\right| = \lim_{k\to\infty}\left|\frac{i}{3}\right|^k$

$$= \lim_{k\to\infty}\left(\frac{1}{3}\right)^k = 0$$

$\boxed{\begin{array}{l} r\in\mathbb{R} \\ -1<r<1 \\ \lim_{k\to\infty} r^k = 0 \end{array}}$

So, $\displaystyle\lim_{k\to\infty}\left(\frac{i}{3}\right)^k = 0.$

Thus, $\displaystyle\lim_{k\to\infty} S_k = \lim_{k\to\infty}\frac{1-\left(\frac{i}{3}\right)^k}{1-\frac{i}{3}} = \frac{1-0}{1-\frac{i}{3}}$

$$= \left(\frac{1}{1-\frac{i}{3}}\right)\cdot\left(\frac{1+i/3}{1+i/3}\right) = \frac{1+i/3}{1+\frac{1}{9}} = \frac{9}{10}+\frac{3}{10}i$$

So, $\sum_{n=0}^{\infty} \left(\frac{i}{3}\right)^n$ converges to $\frac{9}{10} + \frac{3}{10} i$.

Note: For proofs and future definitions we will start our sequences and series at 1 to make it simpler. But the results will still hold for ones that start at some no

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Diverge Theorem: Let $\sum_{n=1}^{\infty} a_n$ be a series of complex numbers. If $\sum_{n=1}^{\infty} a_n$ converges, then $\lim_{n\to\infty} a_n = 0$

[Contrapositive: If $\lim_{n\to\infty} a_n \neq 0$, then $\sum_{n=1}^{\infty} a_n$ diverges]

Proof: HW 1 ▨

<u>Theorem:</u> Let $(b_n)_{n=1}^{\infty}$ be a sequence of complex numbers.
Then, $\lim_{n\to\infty} b_n = 0$ iff $\lim_{n\to\infty} |b_n| = 0$

<u>proof:</u>
($\Rightarrow$) Suppose $\lim_{n\to\infty} b_n = 0$.

Let $\varepsilon > 0$.
Then there exists $N > 0$ where if $n \geq N$ then $|b_n - 0| < \varepsilon$.

So, if $n \geq N$ then
$$\big||b_n| - 0\big| = \big||b_n|\big| = |b_n| = |b_n - 0| < \varepsilon$$

So, $\lim_{n\to\infty} |b_n| = 0$.

$(\Leftarrow)$ Suppose $\lim_{n\to\infty} |b_n| = 0$

Let $\varepsilon > 0$.

Since $\lim_{n\to\infty} |b_n| = 0$, there exists $N > 0$ where if $n \geq N$ then $\big| |b_n| - 0 \big| < \varepsilon$.

So, if $n \geq N$ then

$$|b_n - 0| = |b_n| = \big| |b_n| \big| = \big| |b_n| - 0 \big| < \varepsilon.$$

So, $\lim_{n\to\infty} b_n = 0$. ▨

Lemma:

If $z \neq 1$, then

$$1 + z + z^2 + \dots + z^k = \frac{1 - z^{k+1}}{1 - z}$$

proof: Suppose $z \neq 1$.

Let $S_k = 1 + z + z^2 + \dots + z^k$.

Then,

$$S_k - z \cdot S_k = 1 + z + z^2 + \dots + z^k - z - z^2 - \dots - z^k - z^{k+1}$$

$$= 1 - z^{k+1}$$

Thus, $(1 - z) S_k = 1 - z^{k+1}$.

So $S_k = \dfrac{1 - z^{k+1}}{1 - z}$. $\blacksquare$

Ex: (Geometric Series)

Let $z \in \mathbb{C}$.

Consider the series

$$\sum_{n=0}^{\infty} z^n = 1 + z + z^2 + z^3 + z^4 + \cdots$$

For what z does this series converge?

The partial sums are

$$S_1 = 1$$
$$S_2 = 1 + z$$
$$S_3 = 1 + z + z^2$$
$$\vdots \qquad \vdots$$
$$S_k = 1 + z + z^2 + \cdots + z^{k-1}$$

We have

$$S_k = \begin{cases} k & \text{if } z = 1 \\ \dfrac{1 - z^k}{1 - z} & \text{if } z \neq 1 \end{cases}$$

Case 1: Suppose $|z|<1$

Then, $\lim_{k\to\infty} |z^k| = \lim_{k\to\infty} |z|^k = 0$

$0 \leq |z| < 1$

So, by the previous thm, $\lim_{k\to\infty} z^k = 0$

Thus,

$$\lim_{k\to\infty} S_k = \lim_{k\to\infty} \frac{1-z^k}{1-z} = \frac{1-0}{1-z} = \frac{1}{1-z}$$

So, if $|z|<1$, then $\sum_{n=0}^{\infty} z^n$ converges and $\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$

Case 2: Suppose $|z|>1$.

Then, $\lim_{n\to\infty} |z^n| = \lim_{n\to\infty} |z|^n = \infty$

because $|z| \in \mathbb{R}$ with $|z|>1$

Then, $\lim_{n\to\infty} z^n \neq 0$ [because $\lim_{n\to\infty} |z^n| \neq 0$]

By the divergence test, $\sum_{n=0}^{\infty} z^n$ diverges.

Case 3: Suppose $|z|=1$

Then, $\lim_{n\to\infty} |z^n| = \lim_{n\to\infty} |z|^n = \lim_{n\to\infty} 1^n = 1$

So, $\lim_{n\to\infty} z^n \neq 0$. [because $\lim_{n\to\infty} |z^n| \neq 0$]

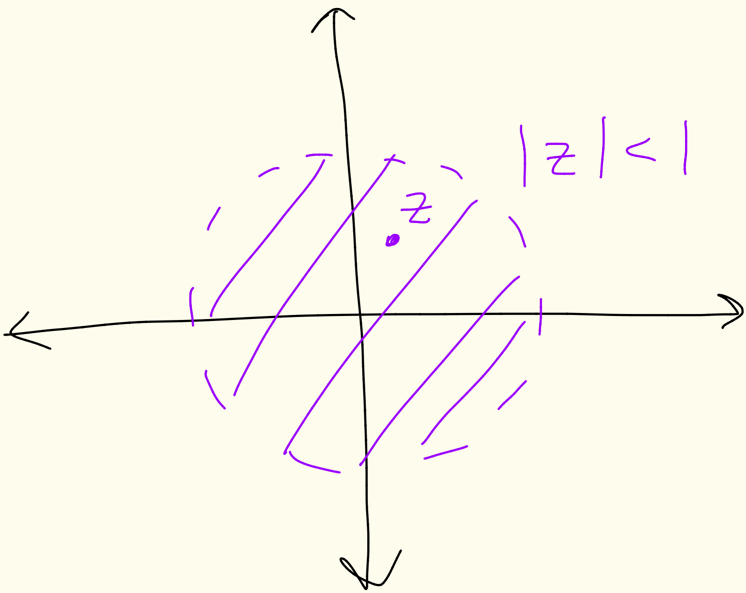
By the divergence test, $\sum_{n=0}^{\infty} z^n$ diverges.

Conclusion:

$\sum_{n=0}^{\infty} z^n$ converges iff $|z|<1$

and if $|z|<1$ then

$$\sum_{n=0}^{\infty} z^n = \frac{1}{1-z}$$



Ex: Consider the p-series

$$\sum_{n=1}^{\infty} \frac{1}{n^p} = \frac{1}{1^p} + \frac{1}{2^p} + \frac{1}{3^p} + \frac{1}{4^p} + \cdots$$

where $p \in \mathbb{R}$

One can show that $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges iff $p > 1$.

Proof: We will prove $(\Leftarrow)$.
The other other direction is in HW 1.
Suppose $p > 1$, $p \in \mathbb{R}$.

We have

$$S_k = \frac{1}{1^p} + \frac{1}{2^p} + \cdots + \frac{1}{k^p}$$

are the partial sums.

Note that the sequence (s_k) is

$$\frac{1}{1^p}, \quad \frac{1}{1^p}+\frac{1}{2^p}, \quad \frac{1}{1^p}+\frac{1}{2^p}+\frac{1}{3^p}, \ldots$$

is an <u>increasing</u> sequence of positive real numbers.

$2^1-1=1$ $\qquad$ $2^2-1=3$

Note that

$2^3-1=7$

$$S_{2^k-1} = \frac{1}{1^p} + \left(\frac{1}{2^p}+\frac{1}{3^p}\right) + \left(\frac{1}{4^p}+\frac{1}{5^p}+\frac{1}{6^p}+\frac{1}{7^p}\right) + \ldots + \left(\frac{1}{(2^{k-1})^p}+\ldots+\frac{1}{(2^k-1)^p}\right)$$

$$\leq \frac{1}{1^p} + \overbrace{\left(\frac{1}{2^p}+\frac{1}{2^p}\right)}^{2 \text{ terms}} + \overbrace{\left(\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}+\frac{1}{4^p}\right)}^{4 \text{ terms}} + \ldots + \overbrace{\left(\frac{1}{(2^{k-1})^p}+\ldots+\frac{1}{(2^{k-1})^p}\right)}^{2^{k-1} \text{ terms}}$$

$$= \frac{1}{1^p} + \frac{2}{2^p} + \frac{4}{4^p} + \cdots + \frac{2^{k-1}}{(2^{k-1})^p}$$

$$= \frac{1}{1^p} + \frac{1}{2^{p-1}} + \frac{1}{4^{p-1}} + \cdots + \frac{1}{(2^{k-1})^{p-1}}$$

$$= \left(\frac{1}{2^{p-1}}\right)^0 + \left(\frac{1}{2^{p-1}}\right)^1 + \left(\frac{1}{2^{p-1}}\right)^2 + \cdots + \left(\frac{1}{2^{p-1}}\right)^{k-1}$$

$$\leq \sum_{n=0}^{\infty} \left(\frac{1}{2^{p-1}}\right)^n = \frac{1}{1 - \frac{1}{2^{p-1}}}.$$

Therefore,

$$S_{2^k - 1} \leq \frac{1}{1 - \frac{1}{2^{p-1}}} \quad \text{for all } k$$

Consider some s_ℓ for some ℓ.

Then, $\ell \leq 2^\ell - 1$.

Thus,

$$s_\ell \leq s_{2^\ell - 1} \leq \frac{1}{1 - \frac{1}{2^{p-1}}}.$$

(s_n) is an increasing sequence

Thus, $(s_\ell)_{\ell=1}^{\infty}$ is an increasing, bounded sequence in $\mathbb{R}$.

By the monotone convergence theorem in 4650, $(s_\ell)_{\ell=1}^{\infty}$ must converge.

Thus, $\sum_{n=1}^{\infty} \frac{1}{n^p}$ converges when $p > 1$.

($\Rightarrow$) HW

Def: Let $\sum_{n=1}^{\infty} a_n$ be a series of complex numbers.
We say that $\sum_{n=1}^{\infty} a_n$ is <u>absolutely convergent</u> if $\sum_{n=1}^{\infty} |a_n|$ converges.

Ex: Consider $\sum_{n=1}^{\infty} \frac{i^n}{n^7}$

Then, $\sum_{n=1}^{\infty} \left|\frac{i^n}{n^7}\right| = \sum_{n=1}^{\infty} \frac{|i|^n}{n^7} = \sum_{n=1}^{\infty} \frac{1}{n^7}$

$\boxed{\left|\frac{a}{b}\right| = \frac{|a|}{|b|} \text{ and } |a^n| = |a|^n}$

which converges since its a $p=7$ series and $7>1$. So, $\sum_{n=1}^{\infty} \frac{i^n}{n^7}$ converges absolutely.

<u>Theorem:</u> If $\sum_{n=1}^{\infty} a_n$ converges absolutely, then $\sum_{n=1}^{\infty} a_n$ converges.

<u>proof:</u> Let

$S_k = \sum_{n=1}^{k} a_n$ be the partial sums of $\sum_{n=1}^{\infty} a_n$

and

$\hat{S}_k = \sum_{n=1}^{k} |a_n|$ be the partial sums of $\sum_{n=1}^{\infty} |a_n|$

We are assuming $\sum_{n=1}^{\infty} |a_n|$ converges, that is $(\hat{S}_k)_{k=1}^{\infty}$ converges.

Let $\varepsilon > 0$.

Since $(\hat{S}_k)_{k=1}^{\infty}$ converges, it is a Cauchy sequence.

So there exists $N > 0$ where if $n \geq m \geq N$, then $|\hat{S}_n - \hat{S}_m| < \varepsilon$.

We now show $(S_n)_{n=1}^{\infty}$ is Cauchy, and so converges.

Let $n \geq m \geq N$.

If $n = m$, then $|S_n - S_m| = |S_n - S_n| = 0 < \varepsilon$

If $n > m$, then

$$|S_n - S_m| = \left| \sum_{k=1}^{n} a_k - \sum_{k=1}^{m} a_k \right|$$

$$= \left| \sum_{k=m+1}^{n} a_k \right| \overset{\boxed{\Delta}}{\leq} \sum_{k=m+1}^{n} |a_k|$$

$$= \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k|$$

$$= \left| \sum_{k=1}^{n} |a_k| - \sum_{k=1}^{m} |a_k| \right|$$

$$= |\hat{S}_n - \hat{S}_m| < \varepsilon$$

So, $(S_n)_{n=1}^{\infty}$ is Cauchy and thus $\sum_{n=1}^{\infty} a_n$ converges $\blacksquare$

Ex: Since $\sum_{n=1}^{\infty} \frac{i^{-n}}{n^7}$ converges absolutely, it also converges.

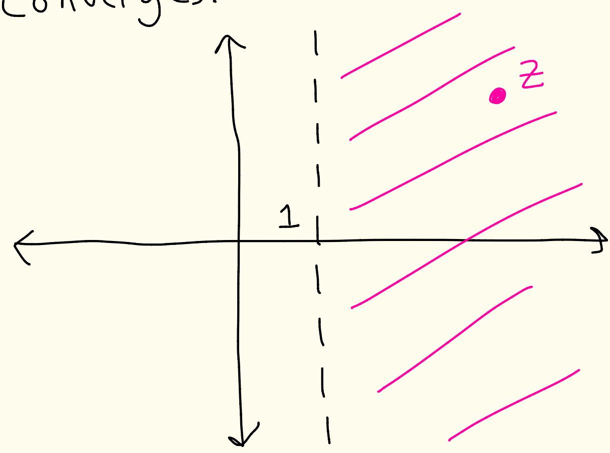
Ex: (Riemann zeta function)

Consider the series

$$\zeta(z) = \sum_{n=1}^{\infty} \frac{1}{n^z} = \frac{1}{1^z} + \frac{1}{2^z} + \frac{1}{3^z} + \cdots$$

where $z \in \mathbb{C}$.

We will show that if $\mathrm{Re}(z) > 1$ then $\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges.

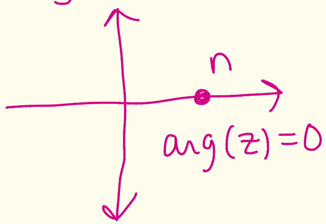


<u>proof:</u> Let $z = x + iy$ with $x = \mathrm{Re}(z) > 1$.

Note that

$$\left|\frac{1}{n^z}\right| = \left|n^{-z}\right| = \left|e^{-z\log(n)}\right|$$

$\log(n) = \ln|n|$



where $\log(z) = \ln|z| + i\arg(z)$
let's choose the branch where $-\pi \le \arg(z) < \pi$

$$= \left|e^{-(x+iy)\ln(n)}\right| = \left|e^{-x\ln(n)}\, e^{i[-y\ln(n)]}\right|$$

$$= \left|e^{-x\ln(n)}\right| \cdot \left|e^{i[-y\ln(n)]}\right| = \left|\underbrace{e^{-x\ln(n)}}_{>0}\right|$$

$$= e^{-x\ln(n)} =$$

$|e^{i\theta}| = 1$, when $\theta \in \mathbb{R}$

$$= e^{-x\ln(n)} = e^{\ln(n^{-x})}$$

$$= n^{-x} = \frac{1}{n^x}$$

Thus, $\displaystyle\sum_{n=1}^{\infty} \left|\frac{1}{n^z}\right| = \sum_{n=1}^{\infty} \frac{1}{n^x}$

which converges since $x > 1$.

Thus, $\displaystyle\sum_{n=1}^{\infty} \frac{1}{n^z}$ converges absolutely

and thus converges, when

$Re(z) > 1$. $\blacksquare$