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TITLE: **CHROMATIC NUMBERS OF DISTANCE GRAPHS WITH DISTANCE SETS** $\{2, 3, X, Y\}$

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CHROMATIC NUMBERS OF DISTANCE GRAPHS WITH DISTANCE SETS $\{2, 3, X, Y\}$

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ABSTRACT

Chromatic Numbers of Distance Graphs

with Distance Sets $\{2, 3, x, y\}$

By

Aileen Sutedja

A distance graph generated by a given set D of positive integers has the set of integers as its vertex set and any two vertices m and n are adjacent (share a common edge) if the absolute difference of m and n is equal to some element d in the set D. The chromatic number of a distance graph is the minimum number of colors required to color all vertices such that no adjacent vertices are assigned the same color. We study the distance graphs generated by $D = \{2, 3, x, y\}$, where x and y are any positive integers. By obtaining bounds for related parameters, such as the density of sequences with missing differences and the kappa value, we acquire new results and complete the determination of the chromatic numbers for all distance graphs with D of the form $\{2, 3, x, y\}$.

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CHAPTER 1

Introduction

What is the smallest number of colors needed to color all the points on the Euclidean plane such that points of unit distance apart get different colors? This question, known as the Hadwinger–Nelson problem, is one of motivating problems behind the study of distance graphs. To this day, the problem is still unsolved, but our current knowledge has narrowed down the plausible answers to be between four and seven [13, 20].

If the Hadwinger–Nelson problem is reduced to the real line, the answer is trivial since using two colors alternatively on unit half open intervals satisfies the condition. Eggleton, Erdős and Skilton [10] raised the complexity of this problem by introducing a distance set D, where D is a subset of the real line, and asking the minimum number of colors needed such that any pair of points having an absolute difference equal to some element of D are assigned different colors. One problem studied in [10] was when the elements of D are positive integers. For such sets D, by isomorphism of components, it is sufficient to study the subgraph induced by the set of integers $\mathbb{Z}$ as the vertex set. This subgraph is known as the integral distance graph.

Given a set D of positive integers, an integral distance graph $G(D)$, or simply a distance graph, is a graph having the set of integers $\mathbb{Z}$ as its vertex set, and two vertices m and n are adjacent (connected by an edge) if $|m - n| \in D$. The focus of the study of distance graphs is to determine the minimum number of colors required so that no two adjacent vertices in the graph have the same color. This minimum number is known as the chromatic number of the graph and is denoted by $\chi(D)$.

For this thesis, we study the 4-element sets D of positive integers with 2 and 3 in the set. We were inspired by the work done by Kemnitz and Kolberg in [15] where they gave the solution for $D = \{2, 3, x, x + s\}$ for $x \in \mathbb{N}, x > 3$ and $s < 10$. In this work, they applied the theorem of Frobenius to explicitly define proper 3-colorings for certain sets D. Another outstanding work on this family of sets D is done by Voigt and Walther [23]. They showed that the distance graph generated by $D = \{2, 3, x, x + s\}$ where $x \geq s^2 - 6s + 3$ and $s \geq 10$, has chromatic number 3.

Our study used a different approach. We employed two main parameters μ and κ and successfully obtained the complete solution to the problem of finding $\chi(\{2, 3, x, y\})$. As we will later present, the parameter μ refers to the density of sets of forbidden differences and the parameter κ is a parameter related to an intriguing conjecture known as "The Lonely Runner Conjecture."

This thesis is organized as such: we begin with preliminary definitions and statements of pertinent results obtained by other authors. Then, we introduce the two major parameters, μ and κ, giving their formal definitions and their relations to other areas of research. Following that, we present our main results, which are decomposed into a number of theorems with overlapping results. In this Main Results chapter, we show how we used the parameters κ and μ to get upper and lower bounds for the chromatic number χ and we also present the algorithms we developed to compute critical bounds for the parameters for some values of x and y. At the end, we consolidate the results and organize them into tabular form for easy reference.

CHAPTER 2

Definitions and Known Results

There are two parts to this chapter. In Section 2.1, we introduce the notion of a graph and graph coloring. Various examples and figures are used to illustrate some concepts. In Section 2.2, we state results on chromatic numbers of distance graphs obtained by other authors, some of which will be used to build our arguments in the subsequent chapters.

2.1 Basic Terminology

Definition 2.1.1. A *graph* $G = (V, E)$ consists of a vertex set V and an edge set E. The elements of V are called *vertices* (or *nodes*), while the elements of E are called *edges* and each edge is an unordered pair of distinct vertices of G.

In this thesis, we study one class of graphs where the vertex set and the edge set are described in the following definition:

Definition 2.1.2. Given a set of positive integers D, the graph $G(D)$ is called an *integral distance graph* (or simply a *distance graph*) generated by the *distance set* D if $V(G) = \mathbb{Z}$ and $E(G) = \{mn : m, n \in V, |m - n| \in D\}$.

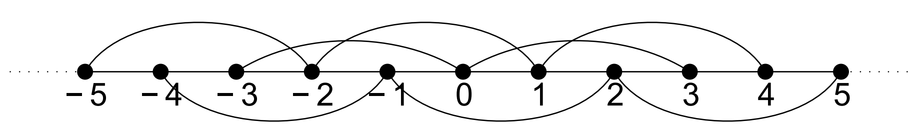

Figure 2.1: An example of a distance graph with $D = \{1, 3\}$

Definition 2.1.3. Two vertices u and v are said to be *adjacent* if $\{uv\} \in E$.

Definition 2.1.4. A *subgraph* of a graph G is a graph H such that $V(H) \subset V(G)$ and $E(H) \subset E(G)$. In particular, if $D' \subset D$, then the distance graph $G(D')$ is a subgraph of the distance graph $G(D)$.

Definition 2.1.5. A *path* $P = \{u_0, \ldots, u_k\}$ of length k is a sequence of $k+1$ distinct vertices, starting with u_0 and ending with u_k such that consecutive vertices are adjacent.

It is worth emphasizing that the *length* of a path P refers to the number of edges in the path, which may or may not equal to the absolute value of the difference of the end vertices. For instance, the path $P_0 = \{-5, -2, 1, 2, 3\}$ in the distance graph $G(\{1, 3\})$ in Figure 2.1 is of the length four, but the difference between the first and the last vertex in absolute value is $|3 - (-5)| = 8$. On the other hand, the path $P_1 = \{-2, -1, 0, 1, 2\}$ has length four and the difference between the first and the last vertex in absolute value is also $|2 - (-2)| = 4$.

Definition 2.1.6. A *cycle* $C = \{u_0, \ldots, u_{k-1}, u_k\}$ is a path of length k such that $u_0 = u_k$ and $u_i \neq u_j$ for $0 \leq i, j \leq k-1$. Since $u_0 = u_k$, a cycle is also known as a *closed path*. If the length k is an even number, we say that C is an *even* cycle. Likewise, if k is an odd number, we say that C is an *odd* cycle.

Definition 2.1.7. A *k-coloring* of a graph G is a function $f : V(G) \rightarrow \{1, ..., k\}$ from the vertex set to the set of positive integers less than or equal to k.

The graph in Figure 2.1 can be colored by a periodic function f by repeating the pattern a, b, c, d on the vertices. The function f, as shown in Figure 2.2, is a 4-coloring of $G(\{1, 3\})$.

Definition 2.1.8. The coloring f is a *proper* coloring of G if for every pair u, v of

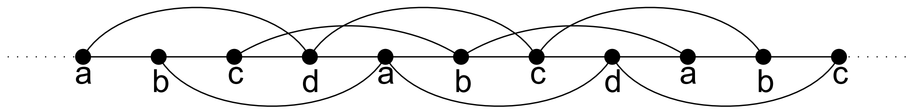

Figure 2.2: A 4-coloring of $G(\{1,3\})$

adjacent vertices, $f(u) \neq f(v)$.

The graph in Figure 2.2 has a proper 4-coloring since no adjacent vertices have the same color. Figure 2.3 shows an improper 3-coloring of the same graph.

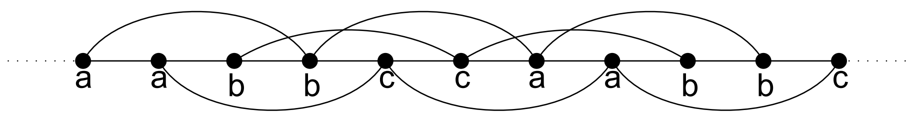

Figure 2.3: An improper coloring of $G(\{1,3\})$

Definition 2.1.9. The *chromatic number* of a graph G, denoted by $\chi(G)$, is the minimum k such that the coloring f is a proper k-coloring. If $\chi(G) = k$ and f is a proper k-coloring, we say that f is a *chromatic coloring*.

Although the function f in Figure 2.2 is a proper 4-coloring of $G(D)$, 4 is not the chromatic number of $G(D)$ since there are other proper colorings of $G(D)$ using less than four colors. One such coloring is shown below.

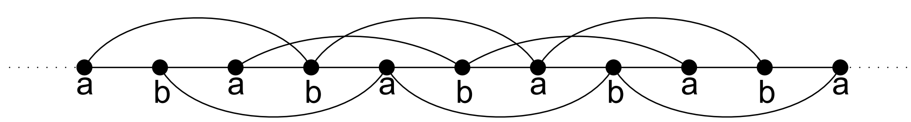

Figure 2.4: A chromatic coloring of $G(\{1,3\})$

The coloring in Figure 2.4 is proper and using two colors is the best we can do for any graph having at least one edge. Hence, the chromatic number of $G(\{1,3\})$ is 2.

Definition 2.1.10. The *floor* of any real number x, denoted by $\lfloor x \rfloor$, is the greatest integer less than or equal to x, and the *ceiling* of any real number x, denoted by $\lceil x \rceil$, is the smallest integer greater than or equal to x.

Definition 2.1.11. Let $x \in \mathbb{R}, d \in \mathbb{N}$. Suppose $x = qd+r$, where $|r| \leq d/2$. Then, $|r|$ is the absolute value of the *absolutely least remainder* of $x \pmod{d}$. This is denoted by $|x|_d$.

For example, let $d = 10$. Then, $|2|_{10} = |8|_{10}$ since $2 = 0 \cdot 10 + 2$ and $8 = 1 \cdot 10 + (-2)$.

Observation 2.1.12. Let $x, y \in \mathbb{R}$ and $d = x + y$. Then $|x|_d = |y|_d$.

Definition 2.1.13. Let $x \in \mathbb{R}$. The minimum distance to an integer function, denoted by $|| * ||$, is defined as: $||x|| := \min\{\lceil x \rceil - x, x - \lfloor x \rfloor\}$.

Definition 2.1.14. Let $S \subset \mathbb{R}$. A number b is called an *upper bound* for S if $x \leq b$, for all $x \in S$. A number b is called the *supremum* of S (also called *least upper bound*), denoted by $\sup S$, if b is the smallest upper bound of S, that is, if c is an upper bound of S, then $b \leq c$.

Similarly, a number b is called a *lower bound* for S if $b \leq x$, for all $x \in S$. A number b is called the *infimum* of S (also called *greatest lower bound*), denoted by $\inf S$, if b is the largest upper bound of S, that is, if c is a lower bound of S, then $c \leq b$.

Let us use one example of S to illustrate the above definition. Let

$$S = \{(-1)^n(1 + 1/n) : n \in \mathbb{N}\} = \{-2, \frac{3}{2}, -\frac{4}{3}, \frac{5}{4}, -\frac{6}{5}, \frac{7}{6}, -\frac{8}{7}, ...\}$$

When n is odd, the n-th term is a negative number between -1 and -2. When n is even, it is a positive number between 1 and $\frac{3}{2}$. Thus, any real number $a \leq -2$ is a lower bound of S and any real number $b \geq \frac{3}{2}$ is an upper bound of S. However, $\inf S$ and $\sup S$ are unique and they are given by $\inf S = -2$ and $\sup S = \frac{3}{2}$.

Definition 2.1.15. Let S be a sequence of real numbers. A point y is called a *cluster point* of S if for any $\varepsilon > 0$ there are infinitely many $x \in S$ with $|x - y| < \varepsilon$.

Definition 2.1.16. Let S be a sequence of real numbers. The *limit superior* of S, denoted by $\limsup S$, is the largest cluster point of S, i.e., the supremum of the set of cluster points. The *limit inferior* of S, denoted by $\liminf S$, is the smallest cluster point of S, i.e., the infimum of the set of cluster points.

Using the same S given in the example after Definition 2.1.14, we have two cluster points, namely -1 and 1. Hence, in this particular example, $\liminf S = -1$ and $\limsup S = 1$.

2.2 Known Results

The appeal of distance graphs has produced substantial results in the determination of the chromatic numbers for various distance sets D. In this section, we state results obtained by other authors and we will also include a proof when the proof is considerably concise.

Let us begin by considering sets D with small cardinalities. When $|D| \leq 2$, we can fairly easily find the chromatic number of the corresponding distance graph $G(D)$.

When D only has one element d, then $\chi(D) = 2$ since

$$f(x) = \begin{cases} a, & \text{for } x \equiv 0, 1, ..., d-1 \pmod{2d}; \\ b, & \text{for } x \equiv d, d+1, ..., 2d-1 \pmod{2d}. \end{cases}$$

is a proper 2-coloring of $G(\{d\})$.

When D has more than one element, we may encounter cases when the elements of D have a common divisor $n > 1$. Lemma 2.2.1 shows that in such a case, the chromatic number of the graph generated by D and that by $D/n = \{d_i/n : d_i \in D\}$ is the same. We can therefore assume without loss of generality that $\gcd(D) = 1$.

Lemma 2.2.1. *[21] Let $D = \{d_1, d_2, ..., d_k\}$. If $n \in \mathbb{N}$, $n|d_\alpha$ for all $d_\alpha \in D$, then $\chi(D) = \chi(D/n)$, where $D/n = \{d_\alpha/n : d_\alpha \in D\}$.*

Before we characterize the chromatic numbers of distance graphs with 2-element sets D, we will prove a more general result for a set D consisting of just odd numbers and a lemma about an odd cycle.

Proposition 2.2.2. *Let $D = \{d_i : d_i$ is an odd integer$\}$. Then $\chi(D) = 2$.*

Proof. The mapping $f : V(G(D)) \to \{a, b\}$ defined by

$$f(x) = \begin{cases} a, & \text{if } x \text{ is even;} \\ b, & \text{if } x \text{ is odd.} \end{cases}$$

is a proper 2-coloring of $G(D)$. □

Lemma 2.2.3. *Let $C = \{u_0, \ldots, u_{2k}, u_{2k+1} = u_0\}$. Then $\chi(C) = 3$.*

Proof. Since $E(C) \neq \emptyset$, $\chi(C) > 1$. The mapping $f : V(C) \to \{a, b, c\}$ defined by

$$f(u_i) = \begin{cases} a, & \text{if } i \text{ is even and } i \neq 0; \\ b, & \text{if } i \text{ is odd and } i \neq 2k+1; \\ c, & \text{if } i = 2k+1. \end{cases}$$

is a proper 3-coloring of $G(C)$.

To complete the proof, we show that there is no proper 2-coloring of C. Let $C_o = \{u_i : i \text{ is odd and } 0 \leq i \leq 2k\}$ and $C_e = \{u_i : i \text{ is even and } 0 \leq i \leq 2k\}$. Then any 2-coloring of C must assign one color to vertices in C_o and another color to vertices in C_e. This coloring cannot be proper since $u_{2k+1} = u_0 \in C_e$ and u_{2k+1} is adjacent to u_{2k} which is also in C_e. Therefore, $\chi(C) = 3$. $\square$

If there is an odd cycle in the graph G, then the chromatic number of G must be at least 3 since using less than three colors would not allow us to properly color the odd cycle. This is indeed true for any subgraph of G. If H is a subgraph of G, then the chromatic number of H is a lower bound for the chromatic number of G since any coloring function with less than $\chi(H)$ colors would not be able to properly color H. Hence, we can make the following observation:

Observation 2.2.4. If H is a subgraph of G, then $\chi(H) \leq \chi(G)$.

If D has finitely many elements, then Lemma 2.2.5 gives an upper bound of $\chi(D)$.

Lemma 2.2.5. *[8] For a finite set D, $\chi(D) \leq |D| + 1$.*

Proof. We color the vertices of the distance graph $G(D)$ recursively with the function $f : \mathbb{Z} \to \mathbb{N}$ as follows. Let $f(0) = 1$. Suppose $f(j)$ is defined for $-i \leq j \leq i$, then we let $f(i+1)$ to be the minimum positive integer not in the set $A = \{f(j) : -i \leq j \leq i \text{ and } i + 1 - j \in D\}$. Next, we let $f(-i-1)$ to be minimum positive integer not in the set $B = \{f(j) : -i \leq j \leq i + 1 \text{ and } j - (-i - 1) \in D\}$. Hence, f is a proper coloring of $G(D)$. Note that the vertex $i+1$ is adjacent to at most $|D|$ smaller vertices, so $|A| \leq |D|$ and similarly, the vertex $-i-1$ is adjacent to at most $|D|$ larger vertices and so, $|B| \leq |D|$. Thus, f is a proper $|D| + 1$ coloring of $G(D)$ and

$\chi(D) \leq |D| + 1.$ □

We now prove that when D consists of two elements, we have the following result:

Proposition 2.2.6. *Let* $|D| = 2, \gcd(D) = 1$. *Then* $\chi(D) = \begin{cases} 2, & \textit{if all } d_i \textit{ are odd;} \\ 3, & \textit{otherwise.} \end{cases}$

Proof. If both elements of D are even, then $\gcd(D) = n > 1$. By Lemma 2.2.1, $\chi(D) = \chi(D/n)$, so it is sufficient to consider the cases when D has two odd elements or an odd and an even element.

If both elements of D are odd, then by Proposition 2.2.2, we have $\chi(D) = 2$.

Now consider the other case when the elements of D are of different parity. Let $D = \{d_o, d_e\}$ where d_o and d_e are odd and even positive integers respectively. We claim that there exists an odd cycle in $G(D)$. Let x be the product of d_o and d_e. Then, there are at least two paths from the integer 0 to the integer x. One path consists of the following vertices $\{0, d_o, 2d_o, \ldots, (d_e - 1) \cdot d_o, d_e \cdot d_o = x\}$ and another consists of $\{0, d_e, 2d_e, \ldots, (d_o - 1) \cdot d_e, d_o \cdot d_e = x\}$. This creates a cycle $\{0, d_0, 2d_o, \ldots, (d_e - 1) \cdot d_o, d_e \cdot d_o = x, d_e \cdot (d_o - 1), \ldots, 2d_e, d_e, 0\}$ of length $d_e + d_o$, which is odd. So, there is an odd cycle in $G(D)$ and thus, by Lemma 2.2.3 and Observation 2.2.4, we have $\chi(D) \geq 3$.

Since $|D| = 2$, by Lemma 2.2.5, $\chi(D) \leq |D|+1 = 3$. Therefore, $\chi(D) = 3$. □

The chromatic number of distance graphs for 3-element sets were studied by Eggleton et al. [10], Chen el al. [8], and Voigt [22], and in 2002, the problem was completely settled by Zhu [27].

Theorem 2.2.7. [27] *Let* $D = \{a, b, c\}$ *with* $a < b < c$ *and* $\gcd(a, b, c) = 1$. *Then*

$$\chi(D) = \begin{cases} 2, & \text{if } a, b, c \text{ are odd;} \\ 4, & \text{if } D = \{1, 2, 3m\} \text{ or } c = a + b \text{ and } b - a \not\equiv 0 \pmod 3; \\ 3, & \text{otherwise.} \end{cases}$$

The next cardinality of D is one that we are most interested in. Due to Kemnitz and Marangio [17, 16], and Liu and Zhu [19], we know that if $D = \{1, 2, 3, 4m\}$ where $m \in \mathbb{N}$, or if $D = \{x, y, y - x, y + x\}$ where x and y are odd integers, then $\chi(D) = 5$. Moreover, Barajas and Serra [1] showed that no other 4-element sets have chromatic number greater than 4.

Theorem 2.2.8. [1] *Let* $|D| = 4$. *Then* $\chi(D) \leq 4$ *unless* $D = \{1, 2, 3, 4m\}$ *where* $m \in \mathbb{N}$, *or* $D = \{x, y, y - x, y + x\}$ *where* x *and* y *are odd integers.*

In our study of sets $D = \{2, 3, x, y\}$, by considering the chromatic number of the subgraph generated by $\{2, 3\}$ as given in Proposition 2.2.6 and by Observation 2.2.4, we have $\chi(D) \geq 3$. Hence, if $\{x, y\}$ is neither $\{1, 4m\}$ nor $\{5, 8\}$, then we have $3 \leq \chi(D) \leq 4$. As mentioned in Chapter 1, we were inspired by Kemnitz and Kolberg's study of $D = \{2, 3, x, x + s\}$ where $x > 3$ and $1 \leq s \leq 9$. They showed that if $\{x, x + s\}$ is not listed in Table 2.1, then $\chi(D) = 3$.

The research done by Eggleton et al. and Voigt and Walther on prime distance sets are also of interest to us. Eggleton et al. [11] proved that if $D \subset \mathbb{P}$ and $D = \{2, 3, x, x + 2\}$ (that is, x and $x+2$ are twin primes), then $\chi(D) = 4$. This result can also be found in Table 2.1. Furthermore, in the following theorem, Voigt and Walther [24] showed that there are only a finite number of prime sets D without twin primes such that $\chi(D) = 4$.

Table 2.1: Sets $D = \{2, 3, x, x + s\}$ with $\chi(D) = 4$ for $1 \leq s \leq 9$ [15]

s	x
1	4, 5, 10
2	$x \not\equiv 2 \pmod 6$
3	$x \not\equiv 3 \pmod 9$, $x \neq 5$
4	5, 6
5	5
6	5
7	4, 5, 6, 10, 11, 12, 16, 17, 22
8	4, 5, 6, 9, 10, 11, 13, 15, 18, 19, 23, 24, 28, 29, 33, 37, 42, 47
9	4, 5, 10

Theorem 2.2.9. *[24] Let $D = \{2, 3, p, q\}$ be a set of primes with $p \geq 7$ and $q > p+2$. Then $\chi(D) = 4$ holds if and only if (p, q) is one of the following:*

$$(11, 19), (11, 23), (11, 37), (11, 41), (17, 29), (23, 31), (23, 41), (29, 37).$$

Voigt and Walther [23] also gave us a result on more general 4-element sets. They proved that $\chi(\{2, 3, x, x+s\}) = 3$ if $s \geq 10$ and $x \geq s^2 - 6s + 3$.

In the following chapters, these existing results will be consolidated with our new results to give the complete solution to the determination of $\chi(\{2, 3, x, y\})$.

CHAPTER 3

The Parameters $\mu(D)$ and $\kappa(D)$

Often times, given a distance graph generated by a fixed distance set D, it is difficult to explicitly define a proper k-coloring and show that any coloring using less than k colors cannot be a proper coloring. Even if such a coloring is found, we may need a completely different coloring when just one element in D is changed. This poses quite a challenge to our current study. The initial attempts to find explicit coloring functions could not be sufficiently adjusted to work for a more general case. It is only when we decided to employ the parameters $\mu(D)$ and $\kappa(D)$ that we could broaden our results. These two parameters allow us to determine the value of $\chi(D)$ by "squeezing" or narrowing the bounds of $\chi(D)$.

3.1 The Parameter $\mu(D)$

Let S be a sequence of non-negative integers. For a non-negative integer n, let $S[n]$ denote the number of elements in S that are less than or equal to n. That is, $S[n] = |S \cap \{0, 1, 2, \ldots, n\}|$. The *upper density* $\overline{\delta}$ and *lower density* $\underline{\delta}$ of S are given by:

$$\overline{\delta}(S) := \limsup_{n\to\infty} \frac{S[n]}{(n+1)} \quad \text{and} \quad \underline{\delta}(S) := \liminf_{n\to\infty} \frac{S[n]}{(n+1)}.$$

If $\overline{\delta}(S) = \underline{\delta}(S)$, then the common value is called the *density* of S, and is denoted by $\delta(S)$. That is,

$$\delta(S) := \lim_{n\to\infty} \frac{S[n]}{(n+1)}.$$

Definition 3.1.1. Let D be a set of positive integers. A sequence S is a *D-sequence* if $s_j - s_k \notin D$ for every $s_j, s_k \in S$.

Alternatively, one can determine whether a sequence S is a D-sequence by looking at the sequence of differences between consecutive elements of S. Let $S = s_0, s_1, s_2, \ldots$ with $s_0 < s_1 < s_2 < \ldots$. Then, the *difference sequence* $\Delta(S)$ is given by $\Delta(S) = \delta_1, \delta_2, \ldots$ where $\delta_i = s_i - s_{i-1}$. By observing that $s_k - s_j = (s_k - s_{k-1}) + (s_{k-1} - s_{k-2}) + \ldots + (s_{j+1} - s_j) = \sum_{i=j+1}^{k} \delta_i$ and $s_k - s_j \notin D$ for every $s_j, s_k \in S$, we get this alternative definition of a D-sequence S:

Definition 3.1.2. A sequence of non-negative integers S is a *D-sequence* if for any indices $j < k$, we have:

$$\sum_{i=j+1}^{k} \delta_i \notin D.$$

Definition 3.1.3. The density of sequences with missing differences in D, denoted by $\mu(D)$, is defined by:

$$\mu(D) := \sup \{\delta(S) : \text{ S is a } D\text{-sequence}\}.$$

The determination of $\mu(D)$ is a question posed by Motzkin in an unpublished collection of problems [6]. This question has largely remained unanswered — getting the exact value is currently possible only when D has no more than two elements [6]. In graph theory, $\mu(D)$ is closely related to the fractional chromatic number of distance graphs.

Definition 3.1.4. The *fractional chromatic number* of a graph G, denoted by $\chi_f(G)$, is the minimum ratio m/n ($m, n \in \mathbb{N}$) of an (m/n)-coloring, where an (m/n)-coloring is a function ϕ from $V(G)$ to n-element subsets of $\{1, 2, \ldots, m\}$ such that if $uv \in E(G)$

then $\phi(u) \cap \phi(v) = \emptyset$.

Further discussion about fractional coloring and fractional chromatic number, including why a minimum ratio exists, can be found in "Algebraic Graph Theory" by Godsil and Royle [12].

Chang, Liu and Zhu [7] proved the following connection between distance graphs and $\mu(D)$:

Lemma 3.1.5. *For any set of positive integers D, the fractional chromatic number of the distance graph generated by D is given by*

$$\chi_f(D) = 1/\mu(D).$$

In the following lemma, Haralambis [14] showed that by studying D-sequences S, we can get an upper bound for $\mu(D)$.

Lemma 3.1.6. [14] *Let D be a set of positive integers, and let $\alpha \in (0, 1]$. If for every D-sequence S with $0 \in S$ there exists a positive integer n such that $S[n]/(n+1) \leqslant \alpha$, then $\mu(D) \leqslant \alpha$.*
In other words, if $\mu(D) > \alpha$, then there exists a D-sequence S with $0 \in S$ such that $S[n]/(n+1) > \alpha$ for any positive integer n.

3.2 The Parameter $\kappa(D)$

In [6], Cantor and Gordon provided a lower bound for $\mu(D)$:

$$\mu(D) \geq \sup_{gcd(t,d)=1} \frac{1}{d} \min_i |td_i|_d.$$

This lower bound is denoted by $\kappa(D)$. We now state equivalent definitions of $\kappa(D)$ as given by Haralambis [14].

Definition 3.2.1. [14] Let D be a finite set of positive integers $D = \{d_i : d_i \in \mathbb{N}\}$.

$$\kappa(D) = \sup_{t \in (0,1)} \min_i ||td_i|| \tag{3.1}$$

$$= \sup_{gcd(t,d)=1} \frac{1}{d} \min_i |td_i|_d \tag{3.2}$$

$$= \max_{\substack{d=d_i+d_j \\ 1 \le t \le d/2}} \frac{1}{d} \min_i |td_i|_d. \tag{3.3}$$

where $|| \cdot ||$ is the minimum distance to an integer function and $|x|_d$ denotes the absolute value of the absolutely least remainder of $x \pmod d$.

Definition 3.2.2. Yet another way of defining $\kappa(D)$ is:

$$\kappa(D) = \sup\{\alpha \in (0, 1/2) : ||td|| \ge \alpha \text{ for some } t \in (0,1), \text{ for all } d \in D\} \tag{3.4}$$

The definition of $\kappa(D)$ in Definition 3.2.1 may make $\kappa(D)$ seem like it is merely a parameter defined to describe a known lower bound for $\mu(D)$. However, this unassuming parameter is involved in a long standing conjecture, popularly known as the "Lonely Runner Conjecture." The conjecture was formulated by Wills [25] in the study of diophantine approximations and independently by Cusick [9] in view-obstruction problems. The unforgettable name is due to Goddyn [4], who contextualized the problem this way:

> Consider k runners running on a circular track of unit length. Each runner runs at a constant speed different from any other runner. A runner is said to be lonely when the distance to the nearest runner is at least $1/k$. The conjecture asserts that for each runner, there exists a time t such that the runner becomes lonely.

This conjecture is usually reformulated in a simpler manner by assuming that the speeds of the runners are integers (see [5]). That is, for any set D of positive integers, there exists t such that $||td|| \geq \frac{1}{k}$, for all $d \in D$, or equivalently, $\kappa(D) \geq \frac{1}{|D|+1}$. The conjecture has been confirmed for $|D| \leq 6$ (up to seven runners) [2, 3, 5], and remains open for $|D| \geq 7$.

It is known that $\chi_f(G) \leq \chi(G)$ holds for all graphs G, and $\chi(D) \leq \lceil 1/\kappa(D) \rceil$ holds for all sets D [18, 26]. Combining these facts with Lemma 3.1.5 we have:

Lemma 3.2.3. [7, 18, 26] *For any given distance set D, it holds that*

$$1/\mu(D) \leq \chi(D) \leq \lceil 1/\kappa(D) \rceil.$$

The following corollary is one that we will often refer to. It illustrates how we use Lemma 3.2.3 in determining $\chi(D)$.

Corollary 3.2.4. *Let D be a set of positive integers. If $\kappa(D) \geq 1/3$, then $\chi(D) \leq 3$; if $\mu(D) < 1/3$, then $\chi(D) \geq 4$.*

In the next chapter, we will see the application of Corollary 3.2.4 in greater detail.

CHAPTER 4

Main Results

At this point, we have every tool we need to accomplish our goal to get the chromatic number of every integer graph having the distance set D of the form $\{2, 3, x, y\}$. We begin with finding the chromatic numbers for some sets D where three out of the four elements are fixed. Then we looked at sets D where x and y differ by a certain constant before we get to the most fascinating part of all, which is finding the chromatic numbers for infinitely many graphs with sufficiently large x and relatively large y. The remaining finitely many sets D are then done individually. At the end, we summarize the established results and our new results in a table.

Let $D = \{2, 3, x, y\}$, $x < y$. From the discussion following Theorem 2.2.8, unless $D = \{1, 2, 3, 4m\}$ or $\{2, 3, 5, 8\}$, we have $3 \leq \chi(D) \leq 4$. If $x = 1$, by Theorems 2.2.7 and 2.2.8, the chromatic number is 5 when y is a multiple of 4; otherwise, the chromatic number is 4.

Henceforth, without loss of generality, for $D = \{2, 3, x, y\}$, we will assume that $4 \leq x < y$.

The following lemma will be useful in the next two theorems in proving that $\mu(D) < 1/3$.

Lemma 4.1.5. *Let $n \in \mathbb{N}$. If $\mu(D) \geq 1/3$, then there exists a D-sequence S such that $S[3t] \geq t + 1$ for all positive integers t such that $3t \leq n$.*

Proof. Let $n \in \mathbb{N}$. Since $\mu(D) \geq 1/3$, we have $\mu(D) > 1/3 - 1/10n$. By Lemma 3.1.6, there exists a D-sequence S such that

$$\frac{S[n]}{n+1} > \frac{1}{3} - \frac{1}{10n},$$

or equivalently,

$$S[n] > \left(\frac{1}{3} - \frac{1}{10n}\right)(n+1).$$

So, for $t \in \mathbb{N}$ and $3t \leq n$, the inequality can be written as

$$S[3t] > \left(\frac{1}{3} - \frac{1}{10n}\right)(3t+1) = t + \frac{1}{3} - \frac{3t+1}{10n}.$$

Note that:

$$\frac{3t+1}{10n} \leq \frac{n+1}{10n} < \frac{1}{3},$$

where the last inequality is obtained from the fact that $n \in \mathbb{N}$. So,

$$t + \frac{1}{3} - \frac{3t+1}{10n} > t \text{ and } S[3t] > t.$$

Since $S[3t]$ is an integer, we can conclude that $S[3t] \geq t+1$ for all $t \in \mathbb{N}$ and $3t \leq n$. $\square$

Theorem 4.1.6. *Let $D = \{2, 3, 6, y\}$, $y \geq 4$. Then*

$$\chi(D) = \begin{cases} 4, & \textit{if } y \equiv 0, \pm 1, \pm 4 \pmod 9; \\ 3, & \textit{otherwise.} \end{cases}$$

Proof. Let $D = \{2, 3, 6, y\}$, $y \geq 4$. Define f by

$$f(z) = \begin{cases} a, & \text{if } z \equiv 0, 1, 5 \pmod 9; \\ b, & \text{if } z \equiv 3, 4, 8 \pmod 9; \\ c, & \text{if } z \equiv 2, 6, 7 \pmod 9. \end{cases}$$

f is a proper 3-coloring for $G(D)$ for $y \not\equiv 0, \pm 1, \pm 4 \pmod 9$. Hence the result follows.

To prove that $\chi(D) = 4$ for the remaining cases, we will use the second part Corollary 3.2.4 and show that if $y \equiv 0, \pm 1, \pm 4 \pmod 9$, then $\mu(D) < 1/3$.

Suppose this is not the case, that is, $\mu(D) \geq 1/3$. Let $y = 9k + r$, where $k \in N$ and $r \in \{0, \pm 1, \pm 4\}$. Let $n = 9(k+1)$. Then, by Lemma 4.1.5, there exists a D-sequence S such that $S[3t] \geq t + 1$, for any $t \leq n/3 = 3(k+1)$.

By substituting various values for t, we figure out the elements of S.

$$t = 1 : S[3] \geq 2 \Rightarrow 0, 1 \in S \text{ (since } 2, 3 \in D);$$

$$t = 2 : S[6] \geq 3 \Rightarrow 5 \in S;$$

$$t = 3 : S[9] \geq 4 \Rightarrow 9 \in S;$$

$$\vdots$$

Continuing this process to $t = 3(k+1)$, we get $\Delta(S) = 1, 4, 4, 1, 4, 4, \ldots, 1, 4, 4 = \delta_1, \delta_2, \delta_3, \delta_4, \delta_5, \delta_6, \ldots, \delta_{3(k+1)-2}, \delta_{3(k+1)-1}, \delta_{3(k+1)} = (1, 4, 4)^{k+1}$.

So,

$$\text{if } r = -4, \text{ then } y = \sum_{i=1}^{3k-1} \delta_i;$$

$$\text{if } r = -1, \text{ then } y = \sum_{i=2}^{3k} \delta_i;$$

$$\text{if } r = 0, \text{ then } y = \sum_{i=1}^{3k} \delta_i;$$

$$\text{if } r = 1, \text{ then } y = \sum_{i=1}^{3k+1} \delta_i;$$

$$\text{if } r = 4, \text{ then } y = \sum_{i=3}^{3(k+1)} \delta_i.$$

Therefore, y is a sum of consecutive elements in $\Delta(S)$, contradicting S being a D-sequence. □

Theorem 4.1.7. *Let $D = \{2, 3, 10, y\}$ or $D = \{2, 3, 4, y\}$, $y \geq 5$. Then*

$$\chi(D) = \begin{cases} 4, & \text{if } y \equiv 0, \pm 1 \pmod 6; \\ 3, & \text{otherwise.} \end{cases}$$

Proof. Let $D = \{2, 3, 10, y\}$ or $D = \{2, 3, 4, y\}$, $y \geq 5$. Define f by

$$f(z) = \begin{cases} a, & \text{if } z \equiv 0, 1 \pmod 6; \\ b, & \text{if } z \equiv 2, 3 \pmod 6; \\ c, & \text{if } z \equiv 4, 5 \pmod 6. \end{cases}$$

f is a proper 3-coloring for $G(D)$ for $y \not\equiv 0, \pm 1 \pmod 6$. Hence the result follows.

To prove that $\chi(D) = 4$ for the remaining cases, we proceed as we did in Theorem 4.1.6 and show that if $y \equiv 0, \pm 1 \pmod 6$, then $\mu(D) < 1/3$.

Suppose this is not the case, that is, $\mu(D) \geq 1/3$. Let $y = 6k + r$, where $k \in N$ and $r \in \{0, \pm 1\}$. Let $n = 6(k + 1)$. Then, by Lemma 4.1.5, there exists a D-sequence S such that $S[3t] \geq t + 1$, for any $t \leq n/3 = 2(k + 1)$.

Let us consider $D = \{2, 3, 10, y\}$. As $2, 3, 10 \in D$, it must be that either $\{0, 1, 5, 6\} \subseteq S$ or $\{0, 1, 6, 7\} \subseteq S$. In either case, by considering the values of t with

$S[3t] \geq t+1$, we conclude that $\Delta(S)$ must be one of the following:

$$\begin{aligned}
\Delta(S_1) &= 1,5,1,5,\dots,1,5 \\
&= \delta_1,\delta_2,\delta_3,\delta_4,\dots,\delta_{2k+1},\delta_{2(k+1)} \\
&= (1,5)^{k+1}, \text{ or} \\
\Delta(S_2) &= 1,4,1,6,1,4,1,6,\dots,1,4,1,6 \\
&= \delta_1,\delta_2,\delta_3,\delta_4,\delta_5,\delta_6,\delta_7,\delta_8,\dots,\delta_{2k-1},\delta_{2k},\delta_{2k+1},\delta_{2(k+1)} \\
&\supset (1,4,1,6)^{\lfloor \frac{k+1}{2} \rfloor}, \text{ or} \\
\Delta(S_3) &= (1,4,1,6)^m(1,5) \text{ for some } m \geq 1 \\
&= 1,4,1,6,\dots,1,5,\dots,1,4,1,6,\dots,1,5 \\
&= \delta_1,\delta_2,\delta_3,\delta_4,\dots,\delta_{4m+1},\delta_{4m+2},\dots, \\
&\quad \delta_{2k+1-4m},\delta_{2k+1-4m+1},\delta_{2k+1-4m+2},\delta_{2k+1-4m+3}\dots,\delta_{2k+1},\delta_{2(k+1)} \\
&\supset [(1,4,1,6)^m(1,5)]^{\lfloor \frac{k+1}{2m+1} \rfloor}.
\end{aligned}$$

We now show that in each case, for $y = 6k+r$, where $k \in N$ and $r \in \{0, \pm 1\}$, y can be written as a sum of consecutive elements in $\Delta(S)$.

Case 1: $S = S_1$.

$$\text{If } r = -1, \text{ then } y = \sum_{i=2}^{2k} \delta_i;$$
$$\text{if } r = 0,1, \text{ then } y = \sum_{i=1}^{2k+r} \delta_i.$$

Case 2: $S = S_2$.

$$\text{If } k \text{ is even and } r = -1, \text{ then } y = \sum_{i=2}^{2k} \delta_i;$$
$$\text{if } k \text{ is even and } r = 0, 1, \text{ then } y = \sum_{i=1}^{2k+r} \delta_i;$$
$$\text{if } k \text{ is odd and } r = -1, \text{ then } y = \sum_{i=2}^{2k+1} \delta_i;$$
$$\text{if } k \text{ is odd and } r = 0, \text{ then } y = \sum_{i=1}^{2k+1} \delta_i;$$
$$\text{if } k \text{ is odd and } r = 1, \text{ then } y = \sum_{i=3}^{2k+2} \delta_i.$$

Case 3: $S = S_3$.

Note that since $\Delta(S_3) \supset [(1,4,1,6)^m(1,5)]^{\lfloor \frac{k+1}{2m+1} \rfloor}$, we have $\sum_i \delta_i \geq 6k + r = y$. Let $q \in \mathbb{Z}$ such that

$$(12m+6)q \leq y < (12m+6)(q+1). \tag{4.1}$$

Taking the first q iterations of $(1,4,1,6)^m(1,5)$, we get a sum of $(12m+6)q$. Now we want to pick the exact number of remaining δ_i's such that the total sum equals y. Subtracting $(12m+6)q$ from Equation (4.1), we get

$$0 \leq 6[k - (2m+1)q] + r < 6(2m+1).$$

If $6[k-(2m+1)q] + r \leq 12m$, then the remaining δ_i's can be picked as in Case 2 (by replacing k with $k-(2m+1)q$). If not, then $12m < 6[k-(2m+1)q] + r < 6(2m+1) = 12m+6$ and $r = \pm 1$.

$$\text{If } r = -1, \text{ then } y = \sum_{i=2}^{(4m+2)(q+1)} \delta_i;$$
$$\text{if } r = 1, \text{ then } y = \sum_{i=1}^{(4m+2)q+4m+1} \delta_i.$$

So in each case, y is a sum of consecutive elements in $\Delta(S)$, contradicting S being a D-sequence.

A similar argument holds for $D = \{2, 3, 4, y\}$, where we get $\Delta(S) = (1, 5) = 1, 5, 1, 5, \ldots.$ Therefore, $\mu(D) < 1/3$ when $y \equiv 0, \pm 1 \pmod 6$ and the result follows. $\square$

Theorem 4.1.8. *Let $D = \{2, 3, x, x + 10\}$, $x \geq 4$. Then*

$$\chi(D) = \begin{cases} 4, & \text{if } x = 5; \\ 3, & \text{otherwise.} \end{cases}$$

Proof. By Theorem 2.2.7, we have $\chi(D) = 4$ when $x = 5$. For $x \neq 5$, it suffices to show that $\kappa(D) \geq 1/3$.

Let $x = 3k + r, k \geq 1, r = 0, 1, 2$ and $n = x + (x + 10) = 2x + 10 = 6k + 2r + 10$. By Equation (3.3) in Definition 3.2.1, one way to show that $\kappa(D) \geq 1/3$ is to find an integer λ such that the following holds for all $d \in D$:

$$\lceil n/3 \rceil \leq |\lambda d|_n \leq n - \lceil n/3 \rceil. \tag{4.2}$$

Let us consider the three cases of x with different values of r.

Case 1: $x = 3k$. Since the case for $x = 6$ is done in Theorem 4.1.6, we can assume $k \geq 3$. We have $\lceil n/3 \rceil = 2k + 4$, so Equation (4.2) becomes:

$$2k + 4 \leq |\lambda d|_n \leq 4k + 6. \tag{4.3}$$

Let A be a set of integers such that for any $a \in A$, we have $|2a|_n$ and $|3a|_n$ satisfying Equation (4.3). It can be easily checked that $A = [k + 2, k + 2 + \lfloor k/3 \rfloor] \subset \mathbb{Z}$ meets our requirement. Then to show $\kappa(D) \geq 1/3$, it is enough to show that there exists some $\lambda \in A$ such that $|\lambda x|_n$ and $|\lambda(x + 10)|_n$ satisfy Equation (4.3).

Since $n = x + (x+10)$, by Observation 2.1.12, $|x|_n = |(x+10)|_n$. This means that an integer λ satisfies Equation (4.3) for x if and only if it does so for $x+10$ as well. Note that as $k \geq 3$, we have $k+2, k+3 \in A$.

If k is even, let $\lambda = k+3$. Then,

$$\begin{aligned}(k+3)x &= (k+3)(3k) \\ &= (3k+9)k \\ &= (6k+10+8)(k/2) \\ &= (6k+10)(k/2) + 4k \\ &\equiv 4k \pmod{n}.\end{aligned}$$

Therefore, if k is even, $\lambda = k+3$ satisfies Equation (4.3).

If k is odd, let $\lambda = k+2$. Then,

$$\begin{aligned}(k+2)x &= (k+2)(3k) \\ &= (k+2)(3(k-1)+3) \\ &= (3k+6)(k-1) + (3k+6) \\ &= (6k+10+2)((k-1)/2) + (3k+6) \\ &= (6k+10)((k-1)/2) + 4k + 5 \\ &\equiv 4k+5 \pmod{n}.\end{aligned}$$

Therefore, if k is odd, $\lambda = k+2$ satisfies Equation (4.3).

Case 2: $x = 3k+1$, for $k \geq 1$. We have $\lceil n/3 \rceil = 2k+4$, so Equation (4.2) becomes:

$$2k+4 \leq |\lambda d|_n \leq 4k+8. \tag{4.4}$$

Let $A = [k+2, k+2+\lfloor (k+2)/3 \rfloor]$. Then, $|2a|_n$ and $|3a|_n$ satisfy Equation (4.4) for any $a \in A$.

As $k \geq 1$, we have $k+2, k+3 \in A$. Similar to Case 1, if $\lambda = k+3$ when k is even and $\lambda = k+2$ when k is odd, then Equation (4.4) is satisfied.

Case 3: $x = 3k+2$, for $k \geq 2$. We have $\lceil n/3 \rceil = 2k+5$, so Equation (4.2) becomes:

$$2k+5 \leq |\lambda d|_n \leq 4k+9. \tag{4.5}$$

Let $A = [k+3, k+3+\lfloor k/3 \rfloor]$. Then, $|2a|_n$ and $|3a|_n$ satisfy Equation (4.5) for any $a \in A$.

Similar to the previous two cases, if $\lambda = k+3$ when k is even and $\lambda = k+4$ when k is odd, then Equation (4.5) is satisfied. □

As mentioned, our methods allow us to find the chromatic numbers for infinitely many distance graphs and by Corollary 3.2.4, to show that the distance graph $G(D)$ has chromatic number 3, it is sufficient to show that $\kappa(D) \geq 1/3$. Hence, we now revisit Definition 3.2.2 to get an idea of a way to obtain a lower bound for $\kappa(D)$.

Let $\theta \in (0, 1/2)$. For a positive integer x, define

$$I_x(\theta) = \{t \in (0,1) : ||tx|| \geq \theta\}.$$

That is,

$$I_x(\theta) = \{t : p + \theta \leq tx \leq p + 1 - \theta, 0 \leq p \leq x-1\}.$$

In Definition 3.2.2, $\kappa(D)$ is defined as:

$$\kappa(D) = \sup\{\theta \in (0, 1/2) : \bigcap_{d \in D} I_d(\theta) \neq \emptyset\}.$$

Therefore, to show that $\kappa(D) \geq 1/3$ for $D = \{2, 3, x, y\}$, it is sufficient to prove that:

$$I_2(1/3) \cap I_3(1/3) \cap I_x(1/3) \cap I_y(1/3) \neq \emptyset.$$

For simplicity, we denote $I_x(1/3)$ by I_x. Each I_x is the union of x disjoint intervals centered at $(2p+1)/2x$, $0 \leq p \leq x-1$, with width $1/3x$. Precisely, we write

$$I_x = \bigcup_{p=0}^{x-1} I_{x,p} = \bigcup_{p=0}^{x-1} \left[\frac{3p+1}{3x}, \frac{3p+2}{3x}\right].$$

We call each $I_{x,p}$ an I_x-*interval*. Notice that the gap between any two consecutive I_x-intervals, $I_{x,p}$ and $I_{x,p+1}$, is $2/(3x)$, twice the length of an I_x-interval.

Since $I_2 \cap I_3 = [1/6, 2/9] \cup [7/9, 5/6]$, by symmetry, to show $\kappa(D) \geq 1/3$ it is enough to show:

$$[1/6, 2/9] \cap I_x \cap I_y \neq \emptyset. \tag{4.6}$$

Theorem 4.1.9. *Let $D = \{2, 3, x, x+s\}$ where $x = 7, 8$ and $s \geq 11$. Then $\chi(D) = 3$.*

Proof. Note that $I_7 \cap [1/6, 2/9] = [4/21, 2/9]$ and $I_8 \cap [1/6, 2/9] = [1/6, 5/24]$. We shall show that Equation (4.6) is satisfied for $s \geq 11$. That is, $I_x \cap I_{x+s} \cap [1/6, 2/9] \neq \emptyset$ for $x = 7, 8$ and $s \geq 11$.

Let $x = 7$. Then, $I_{x+s} \cap [4/21, 2/9] \neq \emptyset$ if

$$\frac{4}{21} \leq \frac{3p+1}{3(7+s)} \leq \frac{2}{9} \quad \text{or} \quad \frac{4}{21} \leq \frac{3q+2}{3(7+s)} \leq \frac{2}{9}$$

for some $0 \leq p, q \leq 6+s$. By rearranging the terms, we can write the first condition as $\frac{63+12s}{63} \leq p \leq \frac{77+14s}{63}$ and since $\left|\frac{(77+14s)-(63+12s)}{63}\right| \geq 1$ when $s \geq 24.5$, there must be some integer value for p when $s \geq 25$.

For $11 \leq s \leq 24$, it can be easily checked that the following p values result in

$I_{x+s,p} \cap [4/21, 2/9] \neq \emptyset$:

$$p = \begin{cases} 3, & \text{if } 11 \leq s \leq 12; \\ 4, & \text{if } 13 \leq s \leq 16; \\ 5, & \text{if } 17 \leq s \leq 21; \\ 6, & \text{if } 22 \leq s \leq 24. \end{cases}$$

Similarly, when $x = 8$, we want to show that $I_{x+s} \cap [1/6, 5/24] \neq \emptyset$. This happens if

$$\frac{1}{6} \leq \frac{3p+1}{3(8+s)} \leq \frac{5}{24} \quad \text{or} \quad \frac{1}{6} \leq \frac{3q+2}{3(8+s)} \leq \frac{5}{24}$$

for some $0 \leq p, q \leq 7+s$. The first condition can be written as $\frac{24+4s}{24} \leq p \leq \frac{32+5s}{24}$ and since $\left|\frac{(32+5s)-(24+4s)}{24}\right| \geq 1$ when $s \geq 16$, there must be some integer value for p when $s \geq 16$.

For $11 \leq s \leq 15$, it can be easily checked that the following p values result in $I_{x+s,p} \cap [1/6, 5/24] \neq \emptyset$:

$$p = \begin{cases} 3, & \text{if } 11 \leq s \leq 12; \\ 4, & \text{if } 13 \leq s \leq 15. \end{cases}$$

Hence, Equation (4.6) is satisfied for $x = 7, 8$ and $s \geq 11$ and so, $\chi(D) = 3$. □

Lemma 4.1.10. *Let $x \geq 12, x \neq 15, 16$. Then there exists some $I_{x,p} \subseteq [1/6, 2/9]$.*

Proof. For $x \geq 12$ and $x \neq 15, 16$, it is straightforward to show that there exists some integer p, $0 \leq p \leq x-1$, such that $x \leq 6p+2$ and $9p+6 \leq 2x$. This implies, $(3p+1)/3x \geq 1/6$ and $(3p+2)/3x \leq 2/9$ and the lemma is proven. □

Lemma 4.1.10 tells us that for $x \geq 12, x \neq 15, 16$, there exists some I_x-interval in $[1/6, 2/9]$. However, for particular cases, it may be useful to know the values of p for which $I_{x,p}$-interval intersects with $[1/6, 2/9]$. So, we developed a way of determining that based on the value of x, as given in the following lemma:

Lemma 4.1.11. *Let p_f and p_l denote the smallest and largest values of p respectively such that $I_{x,p} \cap [1/6, 2/9] \neq \emptyset$ and let k denote the number of complete I_x-intervals in $[1/6, 2/9]$. If $x = 6\alpha + r = 9\beta + t = 18\gamma + v$, where $\alpha, \beta, \gamma \in \mathbb{Z}$ and $0 \leq r \leq 5, -3 \leq t \leq 5, 0 \leq v \leq 17$, then*

$$p_f = \begin{cases} \alpha, & \text{if } r = 0, 1, 2, 3, 4; \\ \alpha + 1, & \text{if } r = 5. \end{cases} \qquad p_l = \begin{cases} 2\beta - 1, & \text{if } -3 \leq t \leq 1; \\ 2\beta, & \text{if } 2 \leq t \leq 5. \end{cases}$$

$$k = \begin{cases} \gamma + 1, & \text{if } v \equiv 8, 12, 13, 14, 17 \pmod{18}; \\ \gamma, & \text{otherwise.} \end{cases}$$

Proof. Let p_f and p_l denote the smallest and largest values of p respectively such that $I_{x,p} \cap [1/6, 2/9] \neq \emptyset$.

To find p_f, we consider two cases – the first case is when only a part of $I_{x,p}$-interval intersects with $[1/6, 2/9]$ and the second is when the entire $I_{x,p}$-interval is within $[1/6, 2/9]$.

Case (p_f.i):

$$\frac{3p_f + 1}{3x} < \frac{1}{6} \leq \frac{3p_f + 2}{3x}$$
$$\frac{x - 4}{6} \leq p_f < \frac{x - 2}{6}.$$

Let $x = 6\alpha + r$, for some $\alpha \in \mathbb{Z}$ and $0 \leq r \leq 5$. Then, the last line can be written as:

$$\alpha + \frac{r-4}{6} \leq p_f < \alpha + \frac{r-2}{6}.$$

So, for $r = 3, 4$, there exists an integer value for p_f, that is, $p_f = \alpha$.

Case (p_f.ii):

$$\frac{1}{6} \leq \frac{3p_f + 2}{3x}$$
$$p_f \geq \frac{x-2}{6}.$$

Again, if we let $x = 6\alpha + r$, for some $\alpha \in \mathbb{Z}$ and $0 \leq r \leq 5$, the last line can be written as:

$$\alpha + \frac{r-2}{6} \leq p_f.$$

Therefore, for $r = 0, 1, 2, 5$, we have $p_f = \alpha + \left\lceil \frac{r-2}{6} \right\rceil = \begin{cases} \alpha, & \text{if } r = 0, 1, 2; \\ \alpha + 1, & \text{if } r = 5. \end{cases}$

Similarly, to find p_l, we consider two cases just as before – the first case being when only a part of $I_{x,p}$-interval intersects with $[1/6, 2/9]$ and the second is when the entire $I_{x,p}$-interval is within $[1/6, 2/9]$.

Case (p_l.i):

$$\frac{3p_l + 1}{3x} \leq \frac{2}{9} < \frac{3p_l + 2}{3x}$$
$$\frac{2x-6}{9} < p_l \leq \frac{2x-3}{9}.$$

Let $x = 9\beta + t$ for some $\beta \in \mathbb{Z}$ and $-3 \leq t \leq 5$, then the last line can be written as:

$$2\beta + \frac{2t-6}{9} < p_l \le 2\beta + \frac{2t-3}{9}.$$

For $t = -3, \pm 2$, there exists an integer value for p_l given by $p_l = \begin{cases} 2\beta - 1, & \text{if } t = -3, -2; \\ 2\beta, & \text{if } t = 2. \end{cases}$

Case (p_l.ii):

$$\frac{3p_l + 2}{3x} \le \frac{2}{9}$$
$$p_l \le \frac{2x-6}{9}.$$

Again, if we let $x = 9\beta + t$ for some $\beta \in \mathbb{Z}$ and $-3 \le t \le 5$, then the last line can be written as:

$$p_l \le 2\beta + \frac{2t-6}{9}.$$

Therefore, for $t \ne -3, \pm 2$, we have $p_l = 2\beta + \left\lfloor \frac{2t-6}{9} \right\rfloor = \begin{cases} 2\beta - 1, & \text{if } t = 0, \pm 1; \\ 2\beta, & \text{if } t = 3, 4, 5. \end{cases}$

It is crucial for $p_f \le p_l$ in order for $I_x \cap [1/6, 2/9] \ne \emptyset$ and this is indeed the case for $x \ge 4, x \ne 5$. Hence, $I_x \cap [1/6, 2/9] \ne \emptyset$ for $x \ge 4, x \ne 5$.

By considering the different cases, it is easy to compute the number of complete I_x-intervals within $[1/6, 2/9]$ for each $x \ge 4, x \ne 5$. If we let $x = 18\gamma + \upsilon$ for some $\gamma \in \mathbb{Z}$ and $0 \le \upsilon \le 17$, and let k denote the number of complete I_x-intervals, then

$$k = \begin{cases} \gamma + 1, & \text{if } \upsilon \equiv 8, 12, 13, 14, 17 \pmod{18}; \\ \gamma, & \text{otherwise.} \end{cases}$$

□

From this point forward, whenever p_f and p_l are used, they refer to the same notations p_f and p_l as defined in Lemma 4.1.11.

Theorem 4.1.12. *Let* $D = \{2, 3, x, y\}$*,* $12 \leq x < y$*,* $x \neq 15, 16$*. Then* $\chi(D) = 3$ *for all* $y \geq 2x$*.*

Proof. By Lemma 4.1.10, there exists an interval $I_{x,p} \subseteq [1/6, 2/9]$. The length of $I_{x,p}$ is $1/(3x)$. Let $y \geq 2x$. Then, the gap between any two consecutive I_y-intervals is $2/(3y)$. Because $2/(3y) \leq 1/(3x)$, each $I_{x,p}$-interval must intersect with some I_y-interval. In particular, the $I_{x,p}$-intervals within $[1/6, 2/9]$ must intersect some I_y-interval. Hence, $I_2 \cap I_3 \cap I_x \cap I_y \neq \emptyset$ and the result follows. □

Theorem 4.1.13. *Let* $D = \{2, 3, x, y\}$*. Then* $\chi(D) = 3$ *for all of the following cases:*

(a) $x = 9$*,* $y \geq 18, y \neq 23$*;*

(b) $x = 11$*,* $y \geq 22, y \neq 23, 27, 28, 32, 37, 41, 46$*;*

(c) $x = 15$*,* $y \geq 26, y \neq 35, 41$*;*

(d) $x = 16$*,* $y \geq 27, y \neq 37$*.*

Proof. As in Theorem 4.1.9, we shall prove this theorem by showing that Equation (4.6) holds for the various pairs of x and y. That is, we wish to show that $I = [1/6, 2/9] \cap I_x \cap I_y$ is nonempty for each x and y.

Case (a): $x = 9$.

Then $I = [1/6, 2/9] \cap I_9 \cap I_y = [1/6, 5/27] \cap I_y$ and $I \neq \emptyset$ if $\dfrac{3p_f + 1}{3y} \leq \dfrac{5}{27}$, or equivalently if $y \geq \dfrac{27p_f + 9}{5}$.

Let $y = 6\alpha + r$, for some $\alpha \in \mathbb{Z}$ and $0 \leq r \leq 5$. By Lemma 4.1.11,

$$p_f = \begin{cases} \alpha + 1, & \text{if } r = 5; \\ \alpha, & \text{otherwise.} \end{cases}$$

When $r \neq 5$,

$$y \geq \frac{27p_f + 9}{5} \iff 6\alpha + r \geq \frac{27\alpha + 9}{5} \iff \alpha \geq 3 - \frac{5}{3}r.$$

Since $\max\{3 - \frac{5}{3}r\} = 3$, which occurs when $r = 0$, then for $y \geq 6 \cdot 3 = 18$ and $y \not\equiv 5 \pmod 6$, we have $y \geq \dfrac{27p_f + 9}{5}$.

When $r = 5$,

$$y \geq \frac{27p_f + 9}{5} \iff 6\alpha + 5 \geq \frac{27(\alpha + 1) + 9}{5} \iff \alpha \geq \frac{11}{3}.$$

This implies that for $y \geq 29$ and $y \equiv 5 \pmod 6$, we have $y \geq \dfrac{27p_f + 9}{5}$. Therefore, for $y \geq 18, y \neq 23$, we have $I \neq \emptyset$ and $\chi(D) = 3$.

Case (b): $x = 11$.

This case is done similarly using p_l instead of p_f. Let $I = [1/6, 2/9] \cap I_{11} \cap I_y = [7/33, 2/9] \cap I_y$ and $I \neq \emptyset$ if $\dfrac{3p_l + 2}{3y} \geq \dfrac{7}{33}$, or equivalently if $y \leq \dfrac{33p_l + 22}{7}$.

Let $y = 9\beta + t$ for some $\beta \in \mathbb{Z}$ and $-3 \leq t \leq 5$. By Lemma 4.1.11,

$$p_l = \begin{cases} 2\beta - 1, & \text{if } -3 \leq t \leq 1; \\ 2\beta, & \text{if } 2 \leq t \leq 5. \end{cases}$$

When $-3 \leq t \leq 1$,

$$y \leq \frac{33p_l + 22}{7} \iff 9\beta + t \leq \frac{33(2\beta - 1) + 22}{7} \iff \beta \geq \frac{11 + 7t}{3}.$$

We can easily verify that the last inequality holds for the following cases:

(1) $t = -3, -2, -1$ and $\beta \geq 2$, that is, $y \geq 15$,

(2) $t = 0$ and $\beta \geq 4$, that is, $y \geq 36$,

(3) $t = 1$ and $\beta \geq 6$, that is, $y \geq 55$.

When $2 \leq t \leq 5$,

$$y \leq \frac{33p_l + 22}{7} \iff 9\beta + t \leq \frac{33(2\beta) + 22}{7} \iff \beta \geq \frac{7t - 22}{3}.$$

For $t = 2, 3, 4$, the last inequality holds if $\beta \geq 2$ and for $t = 5$, the inequality holds if $\beta \geq 5$. So, if $y \geq 22, y \neq 23, 27, 28, 32, 37, 41, 46$, we have $y \leq \dfrac{33p_l + 22}{7}$. Hence, $I \neq \emptyset$ and $\chi(D) = 3$.

Case (c): $x = 15$.

This case is similar to Case (a) where $x = 9$ in that we use p_f to determine whether Equation (4.6) holds. Let $I = [1/6, 2/9] \cap I_{15} \cap I_y = ([1/6, 8/45] \cup \{2/9\}) \cap I_y$ and $I \neq \emptyset$ if $\dfrac{3p_f + 1}{3y} \leq \dfrac{8}{45}$, or equivalently, $y \geq \dfrac{45p_f + 15}{8}$.

It can be easily verified that $I \neq \emptyset$ for the following cases:

(1) $r = 0, \alpha \geq 5$, that is, $y \geq 30$;

(2) $1 \leq r \leq 4, \alpha \geq 3$, that is, $y \geq 19$;

(3) $r = 5, \alpha \geq 7$, that is, $y \geq 47$.

With this, it remains to show that $I \neq \emptyset$ for $y = 29$. When $y = 29$, we have $p_l = 6$ and $\{2/9\} \subset I_{y,6}$ and so, $I \neq \emptyset$. Therefore, for $y \geq 26, y \neq 35, 41$, we have $I \neq \emptyset$ and thus, $\chi(D) = 3$.

Case (d): $x = 16$.

This case is done in the same way we did Case (b) where $x = 11$. Let $I = [1/6, 2/9] \cap I_{16} \cap I_y = (\{1/6\} \cup [5/24, 2/9]) \cap I_y$ and $I \neq \emptyset$ when $\frac{3p_l + 2}{3y} \geq \frac{5}{24}$, or equivalently when, $y \leq \frac{24p_l + 16}{5}$.

Then, $I \neq \emptyset$ for the following cases:

(1) $t \neq 1, \beta \geq 3$, that is, $y \geq 24$;

(2) $t = 1, \beta \geq 5$, that is, $y \geq 46$.

With this, it remains to show that $I \neq \emptyset$ for $y = 28$. When $y = 28$, we have $p_f = 4$ and $\{1/6\} \subset I_{y,4}$ and so, $I \neq \emptyset$. Hence, for $y \geq 27, y \neq 37$, we have $I \neq \emptyset$ and thus, $\chi(D) = 3$. □

The next theorem is central to our success in determining the chromatic numbers of distance graphs generated by $\{2, 3, x, y\}$. It proves that for infinitely many x's, if y is large enough relative to x, then the chromatic number of the distance graph is 3. Before we prove the theorem, note that by Lemma 4.1.11, for a given x, we know the number of I_x-intervals within $[1/6, 2/9]$ and so, we get the following:

Observation 4.1.14. Let γ be a nonnegative integer. If $x \geq 18\gamma - 1$, then γ of the I_x-intervals are within $[1/6, 2/9]$.

Theorem 4.1.15. *If $D = \{2, 3, x, x + s\}$ with $s \geq 11$ and $x \geq 53$, then $\chi(D) = 3$.*

Proof. By Corollary 3.2.4 it is enough to show $\kappa(D) \geq 1/3$, which is equivalent to showing that $I = I_x \cap I_{x+s} \cap [1/6, 2/9] \neq \emptyset$. Assume to the contrary that $I = \emptyset$. Suppose there are exactly k of the I_x-intervals in $[1/6, 2/9]$. Since $x \geq 53 = 18 \cdot 3 - 1$, we have $k \geq 3$. Let a and b be the smallest and the largest I_x-interval values inside

$[1/6, 2/9]$. That is,

$$a = \frac{3p_f + 1}{3x} \qquad \text{and} \qquad b = \frac{3p_l + 2}{3x},$$

where as defined in Lemma 4.1.11, p_f and p_l denote the smallest value and largest values of p such that $I_{x,p} \cap [1/6, 2/9] \neq \emptyset$. Then, the length of the interval $[a, b]$ is $\frac{k + 2(k-1)}{3x} = \frac{3k-2}{3x}$. Since $I = \emptyset$, there must be $k-1$ intervals of I_{x+s} within $[a, b]$, implying that $\frac{k-1+2k}{3(x+s)} = \frac{3k-1}{3(x+s)} > \frac{3k-2}{3x}$. Therefore, $s < \frac{x}{3k-2}$. Given the assumption that $s \geq 11$, we conclude that $x \geq 33k - 21$. Since $k \geq 3$, we get $x \geq 33k - 21 \geq 18(k+1) - 1$ and so, by Observation 4.1.14, there are $k+1$ intervals of I_x within $[1/6, 2/9]$, a contradiction. □

Lemma 4.1.16. *Let $x < y < 2x$. Then there are α I_y-intervals, where $0 < \alpha < 2$, in the gap between every pair of consecutive I_x intervals.*

Proof. Suppose that $\alpha = 0$. Then there is an I_x-gap with no I_y-intervals, which is equivalent to having two or more I_x-intervals in an I_y gap. This implies that $\frac{2}{3x} < \frac{2}{3y} \Leftrightarrow y < x$, a contradiction. Thus, $\alpha > 0$.

Suppose that $\alpha \geq 2$. Then

$$\begin{aligned} & \frac{\alpha}{3y} + \frac{2(\alpha-1)}{3y} < \frac{2}{3x} \\ \Longleftrightarrow \quad & \frac{3\alpha - 2}{3y} < \frac{2}{3x} \\ \Longleftrightarrow \quad & y > \frac{x}{2}(3\alpha - 2) \end{aligned}$$

Since $\alpha \geq 2$, we have $\frac{x}{2}(3\alpha - 2) \geq 2x$ and so, $y > 2x$, a contradiction.

Therefore, in every I_x-gap, there must be at least a portion of an I_y-interval and strictly fewer than two of them. □

Lemma 4.1.17. *Let $D = \{2, 3, x, y\}, 12 \le x < y < 2x, x \ne 15, 16$. Let k denote the number of complete I_x-intervals in $[1/6, 2/9]$ (note that $k \ge 1$). If there are at least $k+1$ I_y-intervals in $[1/6, 2/9]$ and $y \ge \frac{3k+1}{3k-2} \cdot x$, then we have $\chi(D) = 3$.*

Proof. To show that $\chi(D) = 3$, it is sufficient to show $I = [1/6, 2/9] \cap I_x \cap I_y \ne \emptyset$. Suppose $I = \emptyset$. By Lemma 4.1.16, this is only possible if each of the k I_x-intervals lies in the k I_y-gaps. This implies:

$$\frac{k}{3x} + \frac{2(k-1)}{3x} < \frac{2k}{3y} + \frac{(k+1)}{3y}$$

$$\iff \quad \frac{3k-2}{3x} < \frac{3k+1}{3y}$$

$$\iff \quad y < \frac{3k+1}{3k-2} \cdot x,$$

contradicting our assumption that $y \ge \frac{3k+1}{3k-2} \cdot x$. Therefore, $I \ne \emptyset$ and $\chi(D) = 3$. □

Theorem 4.1.18. *Let $D = \{2, 3, x, y\}$, where $x < y < 2x$ and $y = x + s, s \ge 11$. Then $\chi(D) = 3$ for the following cases:*

(a) $x \ge 36$ and $s \ge 40$,

(b) $x = 26$ and $s \ge 20$, i.e. $y \ge 46$,

(c) $x = 30$ and $s \ge 23$, i.e. $y \ge 53$,

(d) $x = 31$ and $s \ge 24$, i.e. $y \ge 55$,

(e) $x = 32$ and $s \ge 24$, i.e. $y \ge 56$,

(f) $x = 35$ and $s \ge 27$, i.e. $y \ge 62$.

Proof. Note that in all of the above cases, the values of x and y differ by at least 18. So, if there is k I_x-intervals in $[1/6, 2/9]$, there must be at least $k+1$ I_y-intervals. We now show that for the above cases, x and y fulfill the conditions of Lemma 4.1.17 and from that, we get our result.

Let $y = x + s$. Rewriting the condition for y in Lemma 4.1.17, we have

$$x + s \geq \frac{3k+1}{3k-2} \cdot x \iff (3k-2)(x+s) \geq (3k+1)x \iff s \geq \frac{3x}{3k-2}.$$

Let $x = 18\gamma + v$, where $\gamma \geq 2$, and $0 \leq v \leq 17$. From Lemma 4.1.11, the number of complete I_x-intervals is given by $k = \begin{cases} \gamma + 1, & \text{if } x \equiv 8, 12, 13, 14, 17 \pmod{18}; \\ \gamma, & \text{otherwise.} \end{cases}$

So, $\dfrac{3x}{3k-2} = \dfrac{3(18\gamma + v)}{3\gamma - 2}$ and this decreases as γ increases. Thus, $\dfrac{3x}{3k-2}$ is highest when $\gamma = 2$ and $v = 17$, i.e. $\dfrac{3x}{3k-2} = \dfrac{3(18\gamma + v)}{3\gamma - 2} = 39.75$.

By the hypothesis of this theorem, we have $s \geq 40$, so $s \geq \dfrac{3x}{3k-2}$ if and only if $y = x + s \geq \dfrac{3k+1}{3k-2} \cdot x$. By Lemma 4.1.17, we conclude that $\chi(D) = 3$.

When $x = 26, 30, 31, 32, 35$, we have $k = 2$. It is easily verified that we have $y = x + s \geq \dfrac{3k+1}{3k-2} \cdot x$ for each x and s given in the hypothesis. Thus, the condition of Lemma 4.1.17 is satisfied and so, $\chi(D) = 3$. □

Lemma 4.1.19. *If* $\dfrac{3k-2}{3k-5} \cdot x \leq y < 2x$, *where* $x \geq 35$ *or* $x = 26, 30, 31, 32$, *then any* k *consecutive* I_y*-intervals intersect some* I_x*-intervals.*

Proof. Suppose, to the contrary, $I_x \cap I_y = \emptyset$. Note that since $y < 2x$, there must be at least $k-1$ I_x-intervals in the length of k I_y-intervals (if there are $k-2$ or fewer, then we get two or more I_y-intervals in an I_x gap, which is impossible when $y < 2x$).

Then,

$$\begin{aligned} \frac{k}{3y} + \frac{2(k-1)}{3y} &> \frac{k-1}{3x} + \frac{2(k-2)}{3x} \\ \iff \qquad y &< \frac{3k-2}{3k-5}x. \end{aligned}$$

This contradicts our hypothesis for y and thus, the lemma is proven. $\square$

Theorem 4.1.20. *If* $D = \{2, 3, x, y\}$, *where* $38 \leq x \leq 52$ *and* $75 \leq y < 2x$, *then* $\chi(D) = 3$.

Proof. To prove the theorem, we shall show that the above values of y satisfy the condition of Lemma 4.1.19, that is, $y \geq \frac{3k-2}{3k-5} \cdot x$. Since $y \geq 75$, there are at least four complete I_y-intervals in [1/6,2/9]. Let k denote the minimum number of consecutive I_y-intervals that guarantees some intersection with an I_x-interval. In other words, we want to show that $k \geq \frac{5y-2x}{3(y-x)}$, which is equivalent to showing that $\frac{3k-2}{3k-5} \cdot x \leq y$.

$$\begin{aligned} \frac{5y-2x}{3(y-x)} &\leq 4 \\ \iff \qquad y &\geq \frac{10}{7}x \\ \Leftarrow \qquad y \geq 75 &\text{ and } 36 \leq x \leq 52. \end{aligned}$$

Therefore, $\frac{3k-2}{3k-5} \cdot x \leq y \leq 2x$, for $38 \leq x \leq 52$ and $y \geq 75$. By Lemma 4.1.19, every consecutive $k \geq 4$ I_y-intervals intersect some I_x-intervals. Since $y \geq 75$, there are at least 4 I_y-intervals in [1/6,2/9] and thus, $[1/6, 2/9] \cap I_x \cap I_y \neq \emptyset$ and $\chi(D) = 3$. $\square$

Using Theorems 4.1.6, 4.1.7, 4.1.8, 4.1.9, 4.1.12, 4.1.13, 4.1.15, 4.1.18 and 4.1.20, we are left with the following to check:

- $x = 9$ and $y = 23$;
- $x = 11$ and $y = 23, 27, 28, 32, 37, 41, 46$;
- $x = 12, 13, 14$ or $17 \leq x \leq 34, x \neq 26, 30, 31, 32$, and $x + 11 \leq y < 2x$.
- $x = 15$ and $y = 35, 41$;
- $x = 16$ and $y = 37$;
- $x = 26$ and $37 \leq y \leq 45$;
- $x = 30$ and $41 \leq y \leq 52$;
- $x = 31$ and $42 \leq y \leq 54$;
- $x = 32$ and $43 \leq y \leq 55$;
- $x = 35$ and $46 \leq y \leq 61$;
- $36 \leq x \leq 52$ and $x + 11 \leq y \leq \min\{2x, x + 40, 75\}$;

Since there are only a finite number of combinations of x and y that we have yet to settle, we used an algorithm similar to that used in Theorem 4.1.13 to determine whether $\kappa(D) \geq 1/3$. The algorithm can be found in Appendix B. We also implement this algorithm on sets $D = \{2, 3, x, x+s\}$ for $10 \leq s \leq 40$ and $4 \leq x \leq s^2 - 6s + 3$. For each s, we listed the values of x for which the algorithm failed to produce $\kappa(D) \geq 1/3$. The list can be found in Appendix C. In the end, we obtain:

Theorem 4.1.21. *Let $D = \{2, 3, x, x + s\}$ with $x \geq 9, x \neq 10$, and $s \geq 11$. Then*

$\kappa(D) \geq 1/3$ *(so* $\chi(D) = 3$*), except* $(x, x+s)$ *falls in the following set:*

$$A = \{(9,23),(11,23),(11,27),(11,28),(11,32),(11,37),(11,41),(11,46),$$

$$(15,35),(15,41),(16,37),(17,29),(18,31),(23,36),(23,41),(24,37),(28,41)\}.$$

For the pairs of $\{x, y\}$ included in A in Theorem 4.1.21, we employed the idea presented in the proofs of Theorems 4.1.6 and 4.1.7 to show that $\chi(D) = 4$. We wrote an algorithm to check the non-existence of a D-sequence S such that $S[3t] \geq t+1$ for sufficiently large t and this algorithm is laid out in Appendix A. Using the algorithm, we confirmed that $\chi(D) = 4$ for all elements in the set A in Theorem 4.1.21, except for $(x, x+s) \in \{(24,37),(28,41)\}$. In the next theorem, we will show that the chromatic number of the graphs generated by $D = \{2,3,x,y\}$, where $\{x,y\}$ is either $\{(24,37),(28,41)\}$, is also 4.

Lemma 4.1.22. *Let* $D = \{2,3,x,y\}$*, where* $x \equiv 0, \pm 1 \pmod 6$ *or* $y \equiv 0, \pm 1 \pmod 6$*, and* f *be a proper* 3*-coloring of* $G(D)$*. Then, there exist three consecutive integers* $z, z+1, z+2$ *that receive different colors, that is,* $|\{f(z), f(z+1), f(z+2)\}| = 3$.

Proof. Suppose no such three consecutive integers exists, then any proper 3-coloring using colors a, b, c must be a periodic function on vertices, repeating the pattern a, a, b, b, c, c (period 6), contradicting the assumption that $x \equiv 0, \pm 1 \pmod 6$ or $y \equiv 0, \pm 1 \pmod 6$. Therefore, there must be some three consecutive integers in $G(D)$ with pairwise-distinct colors. □

Theorem 4.1.23. *If* $D = \{2,3,x,y\}$ *with* $(x,y) \in \{(24,37),(28,41)\}$*, then* $\chi(D) = 4$.

Proof. Let $(x, x+s) = (24,37)$. Suppose to the contrary, $\chi(D) = 3$. Let f be a 3-coloring for $G(D)$ with the colors a, b, c. By Lemma 4.1.22, there exist three consecutive integers with distinct colors. Without loss of generality, we may assume that the three integers are $0, 1, 2$, and that $f(0) = a$, $f(1) = b$ and $f(2) = c$. This implies $f(3) = c$, $f(4) = a$, $f(-1) = a$, $f(-2) = c$, $f(-4) = b$, and $f(6) = b$. Consider the following three cases.

Case 1: $f(32) = a$**.** Then we have the following:

$$\begin{aligned} & f(35) = f(-5) = b, f(30) = f(8) = c \\ \rightarrow\ & f(37) = c, f(-7) = a \\ \rightarrow\ & f(34) = b \\ \rightarrow\ & f(36) = c, f(10) = a \\ \rightarrow\ & f(38) = a, f(12) = f(13) = b \\ \rightarrow\ & f(40) = b, f(14) = c \\ \rightarrow\ & f(16) = a, f(-10) = b \\ \rightarrow\ & f(-8) = f(19) = c \\ \rightarrow\ & f(43) = a \\ \rightarrow\ & f(45) = b \\ \rightarrow\ & f(47) = c \\ \rightarrow\ & f(23) = b \\ \rightarrow\ & f(26) = a \\ \rightarrow\ & \text{impossible to color -11.} \end{aligned}$$

Case 2: $f(32) = b$**.** We have

$$\begin{aligned}
& f(35) = a \\
\to\ & f(33) = f(38) = c \\
\to\ & f(30) = f(9) = a, f(36) = b \\
\to\ & f(27) = b, f(-7) = f(12) = c \\
\to\ & f(24) = c, f(-10) = a \\
\to\ & f(14) = b \\
\to\ & f(17) = a \\
\to\ & f(20) = c, f(15) = b \\
\to\ & f(23) = b, f(18) = a \\
\to\ & \text{impossible to color 21.}
\end{aligned}$$

Case 3: $f(32) = c$**.** We have

$$\begin{aligned}
& f(30) = f(8) = a \\
\to\ & f(33) = f(-7) = c, f(27) = f(5) = b \\
\to\ & f(36) = b, f(24) = c, f(-10) = f(9) = a \\
\to\ & f(39) = a, f(12) = c \\
\to\ & f(42) = c, f(-12) = f(15) = b \\
\to\ & f(-9) = a \\
\to\ & f(-6) = c \\
\to\ & f(-3) = b \\
\to\ & f(21) = a \\
\to\ & \text{impossible to color 18.}
\end{aligned}$$

Therefore, $\chi(\{2, 3, 24, 37\}) = 4$.

Similarly, suppose $\chi(D) = 3$ when $D = \{2, 3, 28, 41\}$. Let f be a 3-coloring for $G(D)$ with colors a, b, c. Without loss of generality, let us assume $f(0) = a$, $f(1) = b$ and $f(2) = c$. This implies $f(3) = c$, $f(4) = a$, $f(-1) = a$, $f(-2) = c$, $f(-4) = b$, and $f(6) = b$. Consider the following three cases.

Case 1: $f(36) = a$**.** Then we have the following:

$$
\begin{aligned}
& f(39) = f(-5) = b, f(8) = f(34) = c \\
\rightarrow\ & f(41) = c, f(11) = f(-7) = a \\
\rightarrow\ & f(13) = f(38) = b, f(9) = c \\
\rightarrow\ & f(10) = a \\
\rightarrow\ & f(12) = b \\
\rightarrow\ & f(40) = f(14) = c \\
\rightarrow\ & f(37) = f(42) = f(16) = a \\
\rightarrow\ & f(44) = b, f(35) = c \\
\rightarrow\ & f(33) = b, f(-6) = a \\
\rightarrow\ & f(5) = a, f(-8) = c \\
\rightarrow\ & f(46) = c \\
\rightarrow\ & f(18) = b \\
\rightarrow\ & f(21) = c \\
\rightarrow\ & f(23) = a \\
\rightarrow\ & \text{impossible to color 20.}
\end{aligned}
$$

Case 2: $f(36) = b$**.** Then we have

$$\begin{aligned} & f(39) = f(-5) = a \\ \rightarrow\quad & f(42) = f(-7) = f(37) = c \\ \rightarrow\quad & f(40) = b, f(34) = f(9) = a \\ \rightarrow\quad & f(43) = a, f(12) = c, f(31) = b \\ \rightarrow\quad & f(15) = b, f(-10) = a \\ \rightarrow\quad & f(-13) = c \\ \rightarrow\quad & \text{impossible to color 28.} \end{aligned}$$

Case 3: $f(36) = c$**.** Then we have

$$\begin{aligned} & f(34) = a \\ \rightarrow\quad & f(37) = f(-7) = c, f(31) = b \\ \rightarrow\quad & f(9) = f(-10) = a, f(40) = b \\ \rightarrow\quad & f(12) = c, f(43) = a \\ \rightarrow\quad & f(15) = b \\ \rightarrow\quad & f(-13) = c \\ \rightarrow\quad & \text{impossible to color 28.} \end{aligned}$$

Therefore, $\chi(\{2, 3, 28, 41\}) = 4$. □

CHAPTER 5

Summary Tables

This last section summarizes the chromatic number of any distance graph with a distance set of the form $\{2, 3, x, y\}$ for any positive integers x and y.

$$\text{If } x = 1, \text{ then } \chi(\{1, 2, 3, y\}) = \begin{cases} 5, & y \equiv 0 \pmod 4; \\ 4, & \text{otherwise.} \end{cases}$$

Barajas and Serra [1], Kemnitz and Marangio [17, 16], and Liu and Zhu [19], proved that for $x \geq 4$, $\chi(D)$ is either 3 or 4, unless $\{x, y\} = \{5, 8\}$, which results in $\chi(\{2, 3, 5, 8\}) = 5$.

Unless listed in the two tables below, the chromatic number of the distance graph with distance set $D = \{2, 3, x, y\}$ where $4 \leq x < y$ and $\{x, y\} \neq \{5, 8\}$ is 3.

Table 5.1: Sets $D = \{2, 3, x, x + s\}$ with $\chi(D) = 4$ for $1 \leq s \leq 10$.

s	x	References
1	4, 5, 10	[15]
2	$x \not\equiv 2 \pmod 6$	[15]
3	$x \not\equiv 3 \pmod 9$, $x \neq 5$	[15]
4	5, 6	[15]
5	5	[15]
6	5	[15]
7	4, 5, 6, 10, 11, 12, 16, 17, 22	[15]
8	4, 5, 6, 9, 10, 11, 13, 15, 18, 19, 23, 24, 28, 29, 33, 37, 42, 47	[15]
9	4, 5, 10	[15]
10	5	Theorem 4.1.8

Table 5.2: Sets $D = \{2, 3, x, y\}$ with $\chi(D) = 4$ for $y \geq x + 11$.

x	y	References
4, 10	$y \equiv 0, \pm 1 \pmod 6$	Theorem 4.1.7
5	all positive integers $y \neq 5$	[1, 16, 19]
6	$y \equiv 0, \pm 1, \pm 4 \pmod 9$	Theorem 4.1.6
$x \geq 7, x \neq 10$	$(x, y) \in \{(9, 23), (11, 23), (11, 27), (11, 28), (11, 32),$ $(11, 37), (11, 41), (11, 46), (15, 35), (15, 41), (16, 37),$ $(17, 29), (18, 31), (23, 36), (23, 41), (24, 37), (28, 41)\}$	new (some in [24])

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APPENDIX A

Algorithm to Show the Non-Existence of a D-sequence S

This algorithm confirms that for certain values of x and y, we cannot find a D-sequence S, where $D = \{2, 3, x, y\}$. This allows us to check the non-existence of such S for some x and y values in Theorems 4.1.6 and 4.1.7.

```
import java.util.*;

public static void main(String[] args) {

Scanner scanner = new Scanner(System.in);

//Prompts the user to enter the values of x and y;

System.out.println("Enter the other two elements of D.");

d1 = 2;

d2 = 3;

d3 = scanner.nextInt();

d4 = scanner.nextInt();


int[] D = {d1, d2, d3, d4};


//initialize S

int[] S = {0,1};

int t_min = 2;
```

```
int t_max = 1000;  //to prevent infinite loop


//Create a big array to contain arrays A_t for various t.
//Send the (t+1)^th row,i.e. row index t, to get possible
//candidates to be added into S.
//This will return an array, which we then store in big array.
int[][] bigArray = new int[t_max+1][];
for(int t = t_min; t < t_max+1; t++){
bigArray[t] = A_t(t, S, d3, d4);
//If this is not empty, we have a candidate to put into S.
int sum = 0;
for (int v : bigArray[t]) { sum += v; }


if(sum != 0 && t<t_max ){
//Expand S and put smallest value of A_t into S.
//Delete that value from big array.
S = expand(S);
int index = nonZeroIndex(bigArray[t]);
S[S.length-1]=bigArray[t][index];
bigArray[t][index] = 0;
}


//There might be a D-sequence S.
```

```
else if(sum != 0 && t==t_max)
{System.out.println("We're still not done at t = "+t_max);}

else{
//retrace to the last/biggest non-zero row t of bigArray,
for(int i = t-1; i > 1; i--){
t=i; int rowSum = 0;
for (int v : bigArray[i]) {rowSum += v;}
if(rowSum != 0){break;}
}

//Keep the first t elements of S, set the (t+1)th element to 0.
S = keepElts(S,t);

//Put the smallest element in bigArray[t] as the (t+1)th element of S
int index = nonZeroIndex(bigArray[t]);
if(index == -1){
index = 0;
System.out.println("There is no such sequence S!");
break;}  \\ <-- This is what we hope to get.
S[t]=bigArray[t][index];
//set the previously occupied holder in bigArray to 0
bigArray[t][index]=0;
```

```
}

}


//This finds the possible candidates to be added into S for a given t.

public int[] A_t(int t, int[] setS, int d3, int d4){

int d1 = 2;

int d2 = 3;

java.util.Arrays.sort(setS);

int maxS = setS[setS.length - 1];


//create arrayList listA_t

ArrayList<Integer> listA = new ArrayList<Integer>();

for (int r=maxS+1; r <= 3*t; r++){

int counter = 0;

for (int i=0; i < setS.length; i++){

int s=setS[i];

if (r - s != d1 && r - s != d2 && r - s != d3 && r - s != d4 )counter++;

}

//If r-s != d for all d in D and all s in S,

//then the counter = no. of elements of S.

//Put this r into the array list

if(counter == setS.length){listA.add(r);}

}
```

```
//Convert arrayList to array before returning.

int[] A_t = new int[listA.size()];

A_t = convertIntegers(listA);

//Return A_t.

return A_t;

}


//Convert arraylist<Integer> to array int.

public static int[] convertIntegers(List<Integer> integers) {

    int[] ret = new int[integers.size()];

    for (int i=0; i < ret.length; i++)

    {ret[i] = integers.get(i).intValue();}

    return ret;

}


//Increase array size by one. This is used to expand S.

public int[] expand(int[] array) {

   int[] temp = new int[array.length+1];

   for (int i = 0; i < array.length; i++) {temp[i] = array[i];}

   return temp;

}


//Keeping the first t elements of S and make S have length t+1.
```

```
public int[] keepElts(int[] array, int t) {
   int[] temp = new int[t+1];
   for (int i = 0; i < t; i++) {temp[i] = array[i];}
   return temp;
}


//Find index of the first nonzero element in a one dim array.
//Note that since our array is already sorted, we don't need to rearrange it.
public int nonZeroIndex(int[] array){
int index = -1;
for(int i=0; i<array.length; i++){if(array[i] != 0){index = i; break;}
}
return index;
}


}
```

APPENDIX B

Algorithm to Find a Lower Bound for Kappa

This algorithm describes the method we use to find a lower bound of κ using the definition given in Equation (3.3) in Definition 3.2.1 .

```
import java.util.Scanner;


public class kappa {

public static void main(String[] args) {

Scanner scanner = new Scanner(System.in);

int d1, d2, d3, d4;


//Prompts the user to enter the values of x and y.

System.out.println("Enter the other two elements of D.");

d1 = 2;

d2 = 3;

d3 = scanner.nextInt();

d4 = scanner.nextInt();


int[] D = {d1, d2, d3, d4};
```

```
//Computes all the possible sums of two elements of D and put them in an array.
int[] bases = possibleBase(D);
java.util.Arrays.sort(bases);


int j = bases.length - 1;
long n = bases[j];
long l = 1;
long k = kap(n, l, d1, d2, d3, d4);


do{
while((3 * k) < n && 2*l < n){
l += 1;
k = kap(n, l, d1, d2, d3, d4);
}


//Checks if the lower bound k we found is greater than n/3.
//If it is, we stop and print out the lambda, n and lower bound k of kappa.
//If it is not, we repeat the process with the next possible value of n.
if(3 * k >= n){
          System.out.println("The multiplier lambda is "+l+",
                  N = "+n+" and kappa >= "+ k+ "/"+n);
          break;
          }
```

```
    else{j -= 1;
if(j>=0){n = bases[j]; l = 1;}
}
} while(j >= 0);

//If we have exhausted all possible values of n and did not find a suitable
    lower bound for kappa, then we stop the process and
    inform the user of the limitation of this method.
if(j<0){
System.out.println("Try another method.");
}

//Returns an array of possible bases n given an array D
public int[] possibleBase(int[] setD){
int[] bases = new int[setD.length * (setD.length - 1) / 2];
int j = setD.length - 1;
int k = j - 1;
for (int i=0;i < bases.length ;i++){
bases[i] = setD[j] + setD[k];
if (k > 0){k -= 1;}
else{j -= 1; k = j - 1;}
}
return bases;
```

```
}

//Returns the smallest absolutely least remainder of lambda d (mod n)
    for particular n and lambda.
public long kap(long n, long l, int...setD){
long[] lambdaD = new long[setD.length];
for(int i=0;i < setD.length; i++){
lambdaD[i] = mod(n, setD[i]*l);
}

java.util.Arrays.sort(lambdaD);
long kap = lambdaD[0];  //the smallest value in the array lambdaD
return kap;
}

//Finds absolutely least remainder of x (mod n)
public long mod(long n, long x){
long xModN = Math.min(x % n,n-(x%n));
return xModN;
}

}
}
```

APPENDIX C

List of Values of x and s Where Algorithm Fails

By running the program described in Appendix B on sets $D = \{2, 3, x, x + s\}$ for $10 \leq s \leq 40$ and $4 \leq x \leq s^2 - 6s + 3$, we find pairs of x and s where the algorithm fails to find a desirable lower bound for κ. The following list is the output we obtained.

```
s   | Values of x

10  | 5

11  | 5, 6

12  | 5, 6, 11, 17

13  | 4, 5, 6, 10, 18, 23, 24, 28

14  | 4, 5, 9, 10

15  | 4, 5, 10

16  | 5, 6, 11

17  | 5, 6, 11

18  | 5, 23

19  | 4, 5, 10

20  | 4, 5, 6, 10, 15

21  | 4, 5, 6, 10, 11, 16

22  | 5, 6

23  | 5
```

```
24  | 5

25  | 4, 5, 6, 10

26  | 4, 5, 6, 10, 11, 15

27  | 4, 5, 10

28  | 5

29  | 5, 6

30  | 5, 6, 11

31  | 4, 5, 6, 10

32  | 4, 5, 10

33  | 4, 5, 10

34  | 5, 6

35  | 5, 6, 11

36  | 5

37  | 4, 5, 10

38  | 4, 5, 6, 10

39  | 4, 5, 6, 10

40  | 5, 6
```