COLORING PRIME DISTANCE GRAPHS

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ABSTRACT

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By

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Let D be a fixed set of prime numbers. In this thesis we investigate the chromatic number of graphs with vertex set of the integers and edges between any pair of vertices whose distance is in D. Such a graph is called a prime distance graph, and the set D is called the distance set. The chromatic number of prime distance graphs is known when the distance set D has at most four primes. In this thesis we begin to classify prime distance graphs with a distance set of five primes. The number theoretic function $\kappa(D)$ is used as a tool, and some general lemmas about $\kappa(D)$ are established.

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Preliminaries

1.1 Distance graphs

In this thesis we will be considering simple graphs, that is, graphs without loops or parallel edges. In this setting a graph can be defined as a pair of sets (V, E), where V can be any set, either finite or infinite, and the set E must be contained in $\{\{v, w\} : v, w \in V, v \neq w\}$. The set V is called the vertex set, and the set E is called the edge set. If $\{v, w\} \in E$, we say that the vertices v and w are adjacent. This is denoted by $v \sim w$.

Let D be a set of positive real numbers, called a distance set, and let $\langle \boldsymbol{X}, d\rangle$ be a metric space. Then the *distance graph on* $\boldsymbol{X}$ *generated by* D, denoted by $G(\boldsymbol{X}, D)$, is the graph with vertex set $\boldsymbol{X}$ and edge set $\{\{x, y\} \subseteq \boldsymbol{X} : d(x, y) \in D\}$. The axioms of a metric space ensure that this is a simple graph. The most famous distance graph studied is the unit distance graph on the Euclidean plane, $G(\boldsymbol{R}^2, \{1\})$ (see [17]). In this thesis we will be primarily interested in integer distance graphs, that is, graphs with the integers, denoted by $\boldsymbol{Z}$, as the vertex and edges between vertices if the absolute value of their difference is in some fixed set D.

The study of integer distance graphs was initiated by Eggleton, Erdős and Skilton [12] in 1985. They investigated integer distance graphs as a simplification to 1 dimension of the 2 dimensional plane unit distance graph. Since then these graphs have been extensively studied [11, 13, 16, 19, 22].

1.2 Coloring

Some of the most interesting questions surrounding distance graphs concern different vertex colorings of the graphs. The most fundamental type of vertex coloring involves assigning each vertex a single color, requiring only that adjacent vertices receive distinct colors. Given a graph $G = (V, E)$ and a set $\mathcal{C}$ of colors, a proper coloring of the vertices of G is a function $c\colon V \to \mathcal{C}$ such that, for every pair of vertices $v, w \in V$, if $v \sim w$, then $c(v) \neq c(w)$. A k-coloring of G is a proper coloring of G such that the set $\{c(v) : v \in V\}$ has k elements. The *chromatic number* of G is the minimum k such that there exists a k-coloring of G. We denote the chromatic number of G by $\chi(G)$. If the underlying space $\boldsymbol{X}$ is understood, we write $\chi(D)$ as an abbreviation of $\chi(G(\boldsymbol{X}, D))$.

A useful, equivalent definition of the chromatic number of a graph involves graph homomorphisms. A graph homomorphism from $G_1 = (V_1, E_1)$ to $G_2 = (V_2, E_2)$ is a function $\phi\colon V_1 \to V_2$ such that, for every pair of vertices $v, w \in V_1$, if $v \sim w$ in G_1, then $\phi(v) \sim \phi(w)$ in G_2. If such a function exists, we say that G_1 is homomorphic to G_2, denoted by $G_1 \to G_2$. The chromatic number of a graph is connected to homomorphisms from G to the complete graph on k vertices, that is, the graph where each verex is adjacent to every other vertex, denoted K_k.

Proposition 1. *For any graph G, $\chi(G) = \min\{k : G \to K_k\}$.*

To see that Proposition 1 is true, note that any homomorphism to K_k can be considered a coloring with the color set defined as the vertex set of K_k, and, conversely, any k-coloring function can be considered a homomorphism to K_k with the vertex set of K_k defined as the set of colors used in the k-coloring.

A proper coloring requires adjacent vertices to receive distinct colors. One might want to strengthen this requirement so that adjacent vertices receive colors that are in some way "far apart." One way to do this is to define the set of colors $\mathcal{C} := [0, 1)$, equipping the interval with the circular distance function $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$. For any $r \in \boldsymbol{R}$, where $\boldsymbol{R}$ is the set of real numbers, an r-circular coloring of a graph G is a function $c\colon V \to [0, 1)$ such that, for every pair of vertices $v, w \in V$, if $v \sim w$, then $d(c(v), c(w)) \geq 1/r$. The *circular chromatic number* of a graph is defined to be the infimum of r over all r-circular colorings of G. The circular chromatic number of G is denoted by $\chi_c(G)$.

The circular chromatic number was introduced by Vince [18]. For a comprehensive survey see [22]. A useful equivalent definition uses graph homomorphisms with the target graph being the circular clique. For a pair of integers n and k such that $n \geq 2k$, let $K_{n/k}$ be the circular clique, defined as the graph with vertex set $\{0, 1, \ldots, n-1\}$ where $i \sim j$ if $i \equiv j + x \pmod{n}$ for some $x \in \{k, k+1, \ldots, n-k\}$.

Proposition 2. *For any graph G, $\chi_c(G) = \inf\{n/k : G \to K_{n/k}\}$.*

From this characterization of the circular chromatic number the following proposition is easy to prove [23].

Proposition 3. *For any graph G, $\chi(G) = \lceil \chi_c(G) \rceil$.*

The idea behind the proof is that if $a/b > c/d$, then $K_{c/d} \to K_{a/b}$. Since $K_{n/1}$ is isomorphic to K_n, the proposition follows.

1.3 The kappa value

For a real number x, let $||x||$ denote the minimum distance from x to an integer, that is $||x|| = \min\{\lceil x \rceil - x, x - \lfloor x \rfloor\}$. For a fixed set D of positive integers and any real t, denote by $||tD||$ the smallest value $||td||$ among all $d \in D$. The *kappa value of* D, denoted by $\kappa(D)$, is the supremum of $||tD||$ among all real t. That is,

$$\kappa(D) := \sup\{||tD|| \colon t \in \boldsymbol{R}\}.$$

The kappa value was introduced (in an alternate form) by Wills [21]. The kappa value is connected to many different questions in diverse fields of mathematics, including diophantine approximations in number theory [5, 6, 7, 8], view obstruction problems in geometry [9] and nowhere zero flows on matroids [3]. Most famously, the kappa value is the subject of the lonely runner conjecture first posed by Wills [21] and given the poetic name by Goddyn [3]:

Conjecture 4. *Let D be a finite set of positive integers. Then $\kappa(D) \geq 1/(|D| + 1)$.*

The conjecture is trivial when $|D| \in \{1, 2\}$ and has been verified when $|D| = 3$ by Betke and Wills [2] in 1972, $|D| = 4$ by Cusick and Pommerance [10] in 1984, $|D| = 5$ by Bohman, Holzman and Kleitman [4] in 2001 and $|D| = 6$ by Barajas and Serra [1] in 2007. The full solution has eluded proof.

Of interest to this thesis, the kappa value can also be used to bound the circular chromatic number of integer distance graphs. To prove this, we use an equivalent definition of the kappa value introduced by Haralambis [15] in order to study the density of integer sequences with missing differences. For an integer x, the notation $|x|_m$ is the circular distance modulo m, that is the minimum $x \pmod{m}$ and $m - (x$

(mod m)). Similar to the definition of $\kappa(D)$,

$$|tD|_m = \min\{|td|_m : d \in D\}.$$

With these notations, we are able to rationalize the kappa value, using integer values for t and looking at the circular distance modulo m instead of modulo 1:

$$\kappa(D) = \sup_{\gcd(t,m)=1} \frac{|tD|_m}{m}.$$

Since, for any finite set D, $\kappa(D)$ is rational, the supremum can be replaced by maximum, ensuring there exists a pair of integers m and t that achieve the kappa value.

Proposition 5. *For any finite set of positive integers D , $\chi_c(D) \leq 1/\kappa(D)$.*

Proof. Assume m and t are relatively prime integers such that $\kappa(D) = |tD|_m/m$. Let $p = |tD|_m$. By definition, this implies that $p \leq td \pmod{m} \leq m - p$ for each $d \in D$.

Claim: The function $\phi\colon \boldsymbol{Z} \to \{0, 1, \ldots, m-1\}$ defined by $\phi(n) = tn \pmod{m}$ is a homomorphism from $G(\boldsymbol{Z}, D)$ to the circular clique $K_{m/p}$.

Let $i \sim j$ in $G(\boldsymbol{Z}, D)$. Without loss of generality, we can assume $i - j = d$ for some $d \in D$. Since $ti - tj \equiv t(i-j) \equiv td \pmod{m}$, by the definition of $K_{m/p}$, $\phi(i) \sim \phi(j)$.

As $\chi_c(D)$ is the infimum of n/k over all homomorphisms from $G(\boldsymbol{Z}, D)$ to $K_{n/k}$, the homomorphism from the claim implies $\chi_c(D) \leq m/p = 1/\kappa(D)$. □

This proposition together with Proposition 3 give the following corollary, which is the main tool used throughout the rest of this thesis.

Corollary 6. *For any finite set of positive integers D, $\chi(D) \leq \lceil 1/\kappa(D) \rceil$.*

CHAPTER 2

Three Lemmas on $\kappa(D)$

In this chapter we will introduce three general lemmas. An alternative definition of κ introduced by Gupta in [14] involves looking at the sets of "good times" for each element $d \in D$, that is, the times $t \in [0,1)$ such that $||td||$ is greater than some desired value. For $\alpha \in [0,1/2]$ and an element $d \in D$, let $I_d(\alpha) = \{t \in [0,1) \colon ||td|| \geq \alpha\}$. Let $I_D(\alpha)$ be the intersection over D of $I_d(\alpha)$. If $I_D(\alpha)$ is not empty, then $\kappa(D) \geq \alpha$. Thus,

$$\kappa(D) = \sup\{\alpha \in [0,1/2] : I_D(\alpha) \neq \emptyset\}.$$

Note that if $\kappa(D) > \alpha$, then $I_D(\alpha)$ is a union of intervals, and if $\kappa(D) = \alpha$, then $I_D(\alpha)$ is a finite union of singletons.

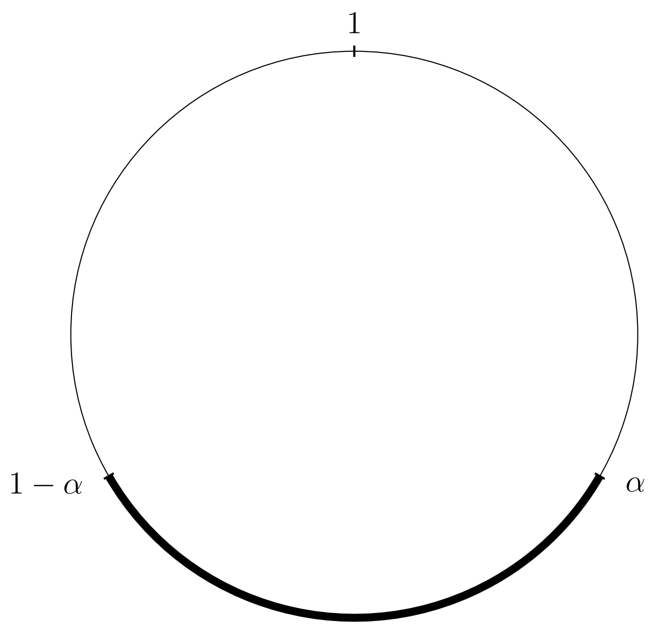

Figure 2.1: The "good times" region for α

2.1 $\kappa(D \cup \{x\})$

If $I_D(\alpha)$ is not empty, one might be interested in how large x must be to guarantee that the intersection of $I_D(\alpha)$ and $I_x(\alpha)$ is not empty. Note that $I_x(\alpha)$ is the union of x disjoint intervals with center $(2n+1)/2x$ for $n \in \{0, 1, \dots, x-1\}$ and width $(1-2\alpha)/x$, that is,

$$I_x(\alpha) = \bigcup_{n=0}^{x-1} \left[\frac{n+\alpha}{x}, \frac{n+1-\alpha}{x}\right].$$

We call these x-intervals. The length of the space between any two consecutive x-intervals is $2\alpha/x$. Now let $[a, b]$ be a connected subset of $I_D(\alpha)$. If the length of the space between each pair of consecutive intervals of $I_x(\alpha)$ is less than the length of that subset, $b - a$, then it must be that one of the intervals of $I_x(\alpha)$ hit the interval $[a, b]$. This can be summarized in the following lemma:

Lemma 7. *Let $[a, b] \subseteq I_D(\alpha)$ with $a < b$. If $x \geq 2\alpha/(b-a)$, then $I_D(\alpha) \cap I_x(\alpha) \neq \emptyset$.*

Lemma 7 is used implicitly throughout the literature on $\kappa(D)$. To my knowledge, the next lemma is new.

2.2 $\kappa(D \cup \{x, x+i\})$

Considering now two elements, we describe an upper bound for the length of an interval of time in which the two sets $I_x(\alpha)$ and $I_{x+i}(\alpha)$ can be disjoint. If this bound is smaller than the length of a target interval contained in $I_D(\alpha)$, we can similarly guarantee that the intersection of $I_D(\alpha)$, $I_x(\alpha)$ and $I_{x+i}(\alpha)$ is not empty.

Lemma 8. *Let $1/4 \leq \alpha \leq 1/3$ and $[a, b] \subseteq I_D(\alpha)$ with $a < b$. If $\frac{4\alpha-1}{i} + \frac{2}{x} \leq b - a$, then $I_D(\alpha) \cap I_x(\alpha) \cap I_{x+i}(\alpha) \neq \emptyset$.*

Proof. We introduce some notation to make it easier to keep track of the different intervals. As noted above,

$$I_x(\alpha) = \bigcup_{n=0}^{x-1} \left[\frac{n+\alpha}{x}, \frac{n+1-\alpha}{x} \right].$$

Fixing α, let $[\frac{n+\alpha}{x}, \frac{n+1-\alpha}{x}]$ be denoted by I_x^n. Let $L(I_x^n)$ be the left endpoint of I_x^n and $R(I_x^n)$ be the right endpoint.

If $i \geq x$, then every x-interval must intersect at least one $(x+i)$-interval, since the length of the gap between $(x+i)$-intervals is less than the length of an x-interval. To show this, consider $\frac{2\alpha}{x+i} \geq \frac{1-2\alpha}{x}$, which simplifies to $i \geq x\,\frac{4\alpha-1}{1-2\alpha}$. When $\alpha = 1/3$, the inequality simplifies to $i \geq x$, and one can check that the right-hand side decreases as α decreases for $1/4 \leq \alpha \leq 1/3$.

In this case, $R(I_x^n) - L(I_x^{n-1}) = \frac{2-2\alpha}{x}$ is an upper bound on the length of an interval during which I_x and I_{x+i} are disjoint. Note that $\frac{2-2\alpha}{x} < \frac{4\alpha-1}{i} + \frac{2}{x}$.

Assume $i < x$ and let I_x^m be any x-interval. If $m = 0$, then with the assumptions on α and i, it can be shown that $L(I_x^0) \leq R(I_{x+i}^0)$, and therefore there is some intersection between the two intervals. If $m \geq 1$, then let I_{x+i}^n be the closest $(x+i)$-interval to I_x^m such that $R(I_{x+i}^n) \leq L(I_x^m)$, and set $L(I_x^m) - R(I_{x+i}^n) = \Delta$. Note that $L(I_x^m) - L(I_x^{m-1}) = 1/x$. This implies that the difference between previous pairs of x and $(x+i)$-intervals decreases until the left point of the x-interval is less than the right point of the $x+i$-interval. More precisely,

$$
\begin{aligned}
L(I_x^{m-r}) - R(I_{x+i}^{n-r}) &= \left(L(I_x^m) - \frac{r}{x}\right) - \left(R(I_{x+i}^n) - \frac{r}{x+i}\right) \\
&= L(I_x^m) - R(I_{x+i}^n) - \frac{r}{x} + \frac{r}{x+i} \\
&= \Delta - \frac{ir}{x(x+i)}.
\end{aligned}
$$

Fix $j \geq 0$ so that $\Delta - \frac{ij}{x(x+i)} \leq 0$ but $\Delta - \frac{i(j-1)}{x(x+i)} > 0$. This implies that

$$
R(I_{x+i}^{n-j}) - L(I_x^{m-j}) = \frac{ij}{x(x+i)} - \Delta \leq \frac{i}{x(x+i)}.
$$

With the assumptions that $i \leq x$ and $\alpha \leq 1/3$, it can be shown that

$$
\frac{i}{x(x+i)} \leq \frac{1-2\alpha}{x} + \frac{1-2\alpha}{x+i}. \tag{2.1}
$$

The right-hand side of the above inequality is the length of an x-interval added to the length of an $(x+i)$-interval. Therefore, since $R(I_{x+i}^{n-j}) - L(I_x^{m-j}) \leq \frac{1-2\alpha}{x} + \frac{1-2\alpha}{x+i}$, there must be some intersection between I_x^{m-j} and I_{x+i}^{n-j}.

Having found an intersection with an x-interval at or before I_x^m, we now move forward, looking at the right endpoint of the x-intervals.

$$
\begin{aligned}
L(I_{x+i}^{n+1+r}) - R(I_x^{m+r}) =& L(I_{x+i}^{n+1}) - R(I_x^m) - \frac{ir}{x(x+i)} \\
=& R(I_{x+i}^n) + \frac{2\alpha}{x+i} - R(I_x^m) - \frac{ir}{x(x+i)} \\
=& L(I_x^m) - \Delta + \frac{2\alpha}{x+i} - R(I_x^m) - \frac{ir}{x(x+i)} \\
=& \frac{2\alpha}{x+i} - \left(\frac{1-2\alpha}{x} + \frac{ir}{x(x+i)} + \Delta\right).
\end{aligned}
$$

Fix $k \geq 0$ so that k is the smallest such that $\frac{2\alpha}{x+i} \leq \frac{1-2\alpha}{x} + \frac{ik}{x(x+i)} + \Delta$, that is, the smallest such that $L(I_{x+i}^{n+1+k}) \leq R(I_x^{m+k})$. We now show that there must be intersection between I_x^{m+k} and I_{x+i}^{n+1+k}. If $k = 0$, then $R(I_{x+i}^{n+1}) > L(I_x^m)$ by our choice of n as the smallest such that $R(I_{x+i}^n) \leq L(I_x^m)$. This, together with the fact that $L(I_{x+i}^{n+1}) \leq R(I_x^m)$ by our choice of k, implies there must be intersection. If $k \geq 1$, then $R(I_x^{m+k-1}) < L(I_{x+i}^{n+k})$. The only way that there is no intersection between I_x^{m+k} and I_{x+i}^{n+1+k} is if the following inequality holds:

$$\begin{aligned} \frac{1-2\alpha}{x} + \frac{1-2\alpha}{x+i} &< R(I_x^{m+k}) - L(I_{x+i}^{n+1+k}) \\ &= R(I_x^{m+k-1}) - L(I_{x+i}^{n+k}) + \frac{i}{x(x+i)} \\ &< \frac{i}{x(x+i)}. \end{aligned}$$

By Eq. (2.1), this contradicts our assumption that $i \leq x$.

In summary, given that $j = \lceil \frac{x(x+i)\Delta}{i} \rceil$ and $k = \lceil \frac{4\alpha x + 2\alpha i - x - i - x(x+i)\Delta}{i} \rceil$, we know that both I_x^{m-j} and I_x^{m+k} intersect I_{x+i}. The length between these gaps is bounded by the following:

$$\begin{aligned} R(I_x^{m+k}) - L(I_x^{m-j}) &= \frac{k+j}{x} + \frac{1-2\alpha}{x} \\ &\leq \frac{\frac{4\alpha x + 2\alpha i - x - i}{i} + 3 - 2\alpha}{x} \\ &= \frac{4\alpha - 1}{i} + \frac{2}{x}. \end{aligned}$$

Note that if $m + k \geq x$, then $R(I_x^{m+k})$ is undefined. In this case the bound $1 - L(I_x^{m-j})$ is smaller than the bound above. Similar arguments apply if $m - j < 0$.

□

Note that $\frac{2}{x}$ is always positive, so, for a fixed small i, if $\frac{4\alpha-1}{i} > b-a$, then the hypothesis of Lemma 8 is not satisfied for any x.

2.3 Rationalizing the good times

The final result of this chapter rationalizes the set of good times by expanding the unit circle to a circle of circumference q. This lemma will be useful because, fixing a rational point and an α, the lemma gives a finite list of residue classes of x modulo q such that the point will be in $I_x(\alpha)$.

Lemma 9. *Fix an integer x and an $\alpha \in (0, 1/2)$, and let p/q be a point in $(0, 1)$. Then $p/q \in I_x(\alpha)$ if and only if $q\alpha \leq xp \pmod{q} \leq q(1-\alpha)$.*

Proof. To say that $p/q \in I_x(\alpha)$ is equivalent to saying that there exists an $n \in \{0, 1, \ldots, x-1\}$ such that $(n+\alpha)/x \leq p/q \leq (n+1-\alpha)/x$. Rearranging this inequality gives $q\alpha \leq px - qn \leq q(1-\alpha)$. □

CHAPTER 3

Prime Distance Graphs

3.1 Introduction

Let $\boldsymbol{P}$ denote the set of prime numbers. In [12] prime distance graphs were considered, that is integer distance graphs with distance set $D \subseteq \boldsymbol{P}$. The first step in the theory of prime distance graphs is to determine the chromatic number of $G(\boldsymbol{Z}, \boldsymbol{P})$.

In the following (in particular see Theorem 12) we will encounter many subgraphs of $G(\boldsymbol{Z}, \boldsymbol{P})$ that are not 3-colorable. The function $c\colon \boldsymbol{Z} \to \boldsymbol{Z}_4$ defined by $c(n) = n \pmod 4$ is a 4-coloring of $G(\boldsymbol{Z}, \boldsymbol{P})$, since if $c(n) = c(m)$, then $n \equiv m \pmod 4$, which implies $|n - m|$ is a multiple of 4, and therefore not prime. This shows that $\chi(\boldsymbol{P}) = 4$.

Thus, since $D \subseteq D'$ implies $\chi(D) \leq \chi(D')$, given that D is a proper subset of $\boldsymbol{P}$, $\chi(D) \in \{1, 2, 3, 4\}$. The task is to classify a set of primes D according to its chromatic number. We say D is class i if $\chi(D) = i$. Clearly the only set that is class 1 is the empty set, and every singleton is class 2. If $|D| \geq 2$, then D is class 2 if and only if $2 \notin D$. Also, if $2 \in D$ but $3 \notin D$, then D is class 3. A less trivial result (see Theorem 12) is that $\{2, 3, p\}$ is class 4 if $p = 5$ and class 3 otherwise. In view of these results, the remaining problem is to classify prime sets $D \supset \{2, 3\}$ with $|D| \geq 4$ into either class 3 or class 4. For a more detailed discussion of these basics of the theory, see [13].

It was shown [13] that if $D = \{2, 3, p, p+2\}$ where p and $p+2$ are twin primes, then D is class 4. Voigt and Walther [19] classified all prime sets with cardinality 4:

Theorem 10. *Let $D = \{2, 3, p, q\}$ be a set of primes with $p \geq 7$ and $q > p+2$. Then D is class 4 if and only if*

$$(p, q) \in \{(11, 19), (11, 23), (11, 37), (11, 41), (17, 29), (23, 31), (23, 41), (29, 37)\}.$$

Since Voigt's paper in 1994, little progress has been made on the subject. In the following chapters we begin to look at prime distance sets with 5 elements that do not contain twin primes or any of the eight minimal class 4 sets of cardinality 4 obtained in Theorem 10. Note that a minimal class 4 set is a set of primes that is class 4 such that no proper subset is class 4.

One interesting question is whether the set of minimal 5 element class 4 sets is finite. In [13] it was shown that

Theorem 11. *The set $\{2, 3\} \cup \{p, p+8, 2p+13\}$ is class 4 whenever p, $p+8$ and $2p+13$ are all primes.*

There is no reason to think that there are only finitely many such sets. Instead we ask a more limited question. Is the set of minimal class 4 sets of the form $\{2, 3, 7, p, q\}$ finite? The results in Chapter 4 almost completely answers this question.

In order to show that a distance set is class 3, we will make extensive use of the kappa value of the distance set. Recall Corollary 6:

$$\chi(D) \leq \left\lceil \frac{1}{\kappa(D)} \right\rceil.$$

Thus, if $\kappa(D) \geq 1/3$, then $\chi(D) \leq 3$. In particular, since we assume $\{2, 3\} \subset D$, if $\kappa(D) \geq 1/3$, then D is class 3.

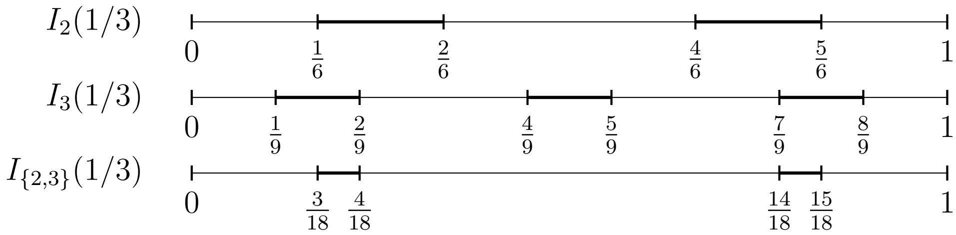

Figure 3.1: The intersection of $I_2(1/3)$ and $I_3(1/3)$

3.2 Known results with new methods

In this section we will recover some of the known results about class 3 sets of three and four primes in order to show how the lemmas in the previous chapter can be applied. Throughout the rest of the chapter, we fix $\alpha = 1/3$, so the notation I_D will be an abbreviation of $I_D(\alpha)$. First we examine sets of three primes.

Theorem 12. *The set $D = \{2, 3, p\}$ is class 4 if $p = 5$ and class 3 otherwise.*

Proof. To show that $\{2, 3, 5\}$ is class 4, we follow the proof from [13]. Consider the two subgraphs of $G(\boldsymbol{Z}, \{2, 3, 5\})$ induced by the vertices $\{0, 2, 3, 5\}$ and $\{1, 3, 4, 6\}$. If they are 3-colorable, then the first forces 2 to be colored the same as 3, and the second graph forces 3 to be colored the same as 4. But this is impossible as 2 is adjacent to 4.

To show that all other three element prime sets are class 3, we will use Lemma 7. First note that, by straightforward calculations, $I_{\{2,3\}} = [3/18, 4/18] \cup [14/18, 15/18]$ (see Fig. 3.1), so the length of a longest interval is 1/18. By Lemma 7, if $p \geq 12$, then $\{2, 3, p\}$ is class 3. The only prime sets left to check are $\{2, 3, 7\}$ and $\{2, 3, 11\}$, which both have kappa value greater than 1/3. □

We now prove a weaker statement than Theorem 10, leaving out the proof that the sets listed are indeed class 4.

Theorem 13. *Let $D = \{2, 3, p, q\}$ be a set of primes with $p \geq 7$ and $q > p+2$. Then D is class 3 if*

$$(p, q) \notin \{(11, 19), (11, 23), (11, 37), (11, 41), (17, 29), (23, 31), (23, 41), (29, 37)\}.$$

Proof. We apply Lemma 8 to $I_{\{2,3\}}$ to find bounds for which the set $\{2, 3, p, p+i\}$ is class 3. As we have seen, the the length of a longest connected interval in that set $I_{\{2,3\}}$ is 1/18. The first step is to find the smallest gap i between primes that allows us to use Lemma 8, that is, for what i does the equation $1/3i < 1/18$ hold. We see that if $i \leq 6$, then the inequality in the hypothesis of Lemma 8 will never hold. Since both p and $p+i$ must be prime, i must be even. Applying Lemma 8 with $i = 8$ gives that if $p \geq 144$ and $i \geq 8$, then the set $\{2, 3, p, p+i\}$ is class 3.

There are 9 pairs of primes p and $p+8$ with $p < 144$:

$$\{11, 19\}, \{23, 31\}, \{29, 37\}, \{53, 61\}, \{59, 67\}, \{71, 79\}, \{89, 97\}, \{101, 107\}, \{131, 139\}.$$

Of these, the only ones with $\kappa(\{2, 3, p, p+8\}) < 1/3$ are $\{11, 19\}, \{23, 31\}, \{29, 37\}$.

Similarly, we can apply Lemma 8 with increasing even gaps i. At each stage we get a bound on how large p must be to guarantee that $\kappa(\{2, 3, p, p+i\}) < 1/3$. By manually checking all prime pairs $(p, p+i)$ less than that bound, we can completely determine which sets $\{2, 3, p, p+i\}$ have kappa value less than 1/3. When $i = 30$ the bound on p can be calculated to be 45. After applying Lemma 8 for all $i \in \{10, 12, \ldots, 30\}$, we can conclude that if $p \geq 45$ and $i \geq 8$, then $\kappa(\{2, 3, p, p+i\}) < 1/3$. The fact that Lemma 8 is used up to $i = 30$ is an arbitrary choice, but at some

Table 3.1: *Applying Lemma 7 to* $\{2,3,p\}$ *for primes* $5 < p < 45$

p	$[a,b] \subset I_{\{2,3,p\}}$	Bound on q	Primes q with $\kappa(\{2,3,p,q\}) < 1/3$
7	$[4/21, 2/9]$	21	5
11	$[7/33, 2/9]$	66	5,13,19,23,37,41
13	$[7/39, 8/39]$	26	5,11
17	$[10/51, 11/51]$	34	5,19,29
19	$[10/57, 11/57]$	38	5,11,17
23	$[13/69, 14/69]$	46	5,11,31,41
29	$[16/87, 17/87]$	58	5,17,31,37
31	$[19/93, 20/93]$	62	5,23,29
37	$[22/111, 23/111]$	74	5,11,29
41	$[25/123, 26/123]$	82	5,11,23,43
43	$[25/129, 26/129]$	86	5,41

point Lemma 7 is needed to establish that the kappa value is less than 1/3 for D sets containing a smaller primes. By applying Lemma 7 to $\{2,3,p\}$ for primes $5 < p < 45$, we can show that that if $i \geq 8$, then the set $\{2,3,p,p+i\}$ is class 3 unless $\{p, p+i\}$ is a pair of primes in the statement of the theorem. See Table 3.1.

To complete the proof we must show that, for $i = 4$ or $i = 6$, the sets $\{2,3,p,p+i\}$ are class 3. To do this we use Lemma 9. We want to show that there exists a rational point in the interval $[3/18, 4/18]$ that is in $I_{\{p,p+4\}}$, given that p is equivalent to some number modulo the denominator of the point. We start by checking the points $\{n/90 : 15 \leq n \leq 20\}$. The choice of denominator 90 is arbitrary, but it is convenient if the end points of the target interval have denominators which divide the denominator of the points checked. Note that since both p and $p+4$ are primes, we know $p \equiv 1 \pmod 6$.

From Table 3.2, we see that if $p \equiv 1 \pmod{90}$, then $p+4$ is not prime, if $p \equiv 85 \pmod{90}$, then p is not prime, and if $p \not\equiv 37, 49 \pmod{90}$, then there exists

Table 3.2: *Rational points in* $I_{\{2,3\}} \cap I_{\{p,p+4\}}$ *(Round 1)*

$p \pmod{90}$	$\gcd(p, 90)$	$\gcd(p+4, 90)$	Point in $I_{\{p,p+4\}}$
1		5	
7			2/9
13			1/5
19			17/90
25	5		2/9
31		5	19/90
37			
43			8/45
49			
55	5		19/90
61		5	2/9
67			17/90
73			1/5
79			2/9
85	5		

Table 3.3: *Rational points in* $I_{\{2,3\}} \cap I_{\{p,p+4\}}$ *(Round 2)*

$p \pmod{180}$	$\gcd(p, 180)$	$\gcd(p+4, 180)$	Point in $I_{\{p,p+4\}}$
37			37/180
49			13/60
127			13/60
139			37/180

a rational point in $[3/18, 4/18]$ contained in $I_{\{p,p+4\}}$. Now we increase the number of points we are checking by a factor of 2 to see if, when $p \equiv 37, 49 \pmod{90}$, there exists a point in $\{n/180 : 30 \leq n \leq 40\}$ contained in $I_{\{p,p+4\}}$.

From Tables 3.2 and 3.3 we see that indeed all prime sets $\{2, 3, p, p+4\}$ are class 3. In a similar way we show that $\{2, 3, p, p+6\}$ is class 3 for all prime pairs p and $p+6$. To start, we again check the points $\{n/90 : 15 \leq n \leq 20\}$, but this time we must check both $p \equiv 1, 5 \pmod{6}$. Tables 3.4 to 3.6 show that all sets of the form

Table 3.4: *Rational points in* $I_{\{2,3\}} \cap I_{\{p,p+6\}}$ *(Round 1)*

$p \pmod{90}$	$\gcd(p, 90)$	$\gcd(p+6, 90)$	Point in $I_{\{p,p+6\}}$
1			
5	5		
7			1/5
11			
13			17/90
17			1/5
19		5	8/45
23			17/90
25	5		8/45
29		5	17/90
31			8/45
35	5		19/90
37			8/45
41			
43			
47			8/45
49		5	19/90
53			8/45
55	5		17/90
59		5	8/45
61			17/90
65	5		8/45
67			1/5
71			17/90
73			
77			1/5
79		5	
83			
85	5		
89		5	

Table 3.5: *Rational points in* $I_{\{2,3\}} \cap I_{\{p,p+6\}}$ *(Round 2)*

$p \pmod{630}$	$\gcd(p, 630)$	$\gcd(p+6, 630)$	Point in $I_{\{p,p+6\}}$
1		7	
11			67/315
41			113/630
43		7	6/35
73			107/630
83			109/630
91	7		107/630
101			109/630
131			6/35
133	7		53/315
163			53/315
173			107/630
181			53/315
191			107/630
221			107/630
223			53/315
253		7	53/315
263			109/630
271			53/315
281		7	109/630
311			
313			
343	7		109/630
353			53/315
361			109/630
371	7		53/315
401			53/315
403			107/630
433			107/630
443			53/315
451			107/630
461			53/315
491		7	53/315
493			6/35
523			109/630
533		7	107/630
541			109/630
551			107/630
581	7		6/35
583			113/630
613			67/315
623	7		

Table 3.6: *Rational points in* $I_{\{2,3\}} \cap I_{\{p,p+6\}}$ *(Round 3)*

$p \pmod{1260}$	$\gcd(p, 1260)$	$\gcd(p+6, 1260)$	Point in $I_{\{p,p+6\}}$
311			71/420
313			211/1260
941			211/1260
943			71/420

$\{2, 3, p, p+6\}$ are class 3.

□

CHAPTER 4

Class 3 Prime Sets of the Form $\{2,3,7,p,q\}$

In this chapter we attempt to emulate the proof of Theorem 13 in order to show that there are only finitely many minimal class 4 prime sets of the form $\{2,3,7,p,q\}$.

4.1 Applying Lemma 8

We first apply Lemma 8, with $\alpha = 1/3$, to obtain bounds for which $\{2,3,7,p,q\}$ is class 3. The interval $[4/21, 2/9] \subset I_{\{2,3,7\}}$, and the length of this interval is $2/63$. The first step is to determine the smallest gap i such that $1/3i < 2/63$, in order to ensure that the inequality in the hypothesis of Lemma 8 can be satisfied. If $i \leq 10$, then $1/3i > 2/63$, so, since p must be prime, the gaps considered are the even integers $i \geq 12$.

Fixing $\boxed{i = 12}$, we solve the following inequality from Lemma 8 for p: $\frac{1}{3i} + \frac{2}{p} \leq \frac{2}{63}$. Thus if $p \geq 504$ and $q \geq p + 12$, then, by Lemma 8, $\{2,3,7,p,q\}$ will be class 3.

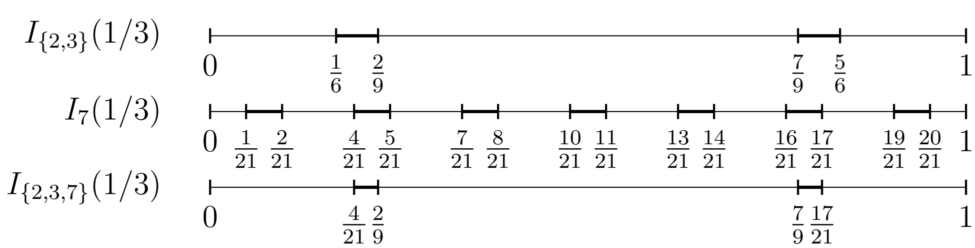

Figure 4.1: The intersection of $I_7(1/3)$ and $I_{\{2,3\}}(1/3)$

There are 47 pairs of primes $(p, p+12)$ such that $p < 504$:

$(5, 17)$	$(7, 19)$	$(11, 23)$	$(17, 29)$	$(19, 31)$	$(29, 41)$	$(31, 43)$
$(41, 53)$	$(47, 59)$	$(59, 71)$	$(61, 73)$	$(67, 79)$	$(71, 83)$	$(89, 101)$
$(97, 109)$	$(101, 113)$	$(127, 139)$	$(137, 149)$	$(139, 151)$	$(151, 163)$	$(167, 179)$
$(179, 191)$	$(181, 193)$	$(199, 211)$	$(211, 223)$	$(227, 239)$	$(229, 241)$	$(239, 251)$
$(251, 263)$	$(257, 269)$	$(269, 281)$	$(271, 283)$	$(281, 293)$	$(337, 349)$	$(347, 359)$
$(367, 379)$	$(389, 401)$	$(397, 409)$	$(409, 421)$	$(419, 431)$	$(421, 433)$	$(431, 443)$
$(449, 461)$	$(467, 479)$	$(479, 491)$	$(487, 499)$	$(491, 503)$.		

Of these, there are only 15 such that the set $D = \{2, 3, 7, p, p+12\}$ has $\kappa(D) < 1/3$:

$(5, 17)$	$(11, 23)$	$(17, 29)$	$(19, 31)$	$(29, 41)$	$(31, 43)$	$(41, 53)$	$(47, 59)$
$(61, 73)$	$(67, 79)$	$(71, 83)$	$(89, 101)$	$(97, 109)$	$(139, 151)$	$(181, 193)$.	

This shows that the set $\{2, 3, 7, p, p+12\}$ is class 3 for all pairs of primes p and $p+12$ except (possibly) for the 15 pairs listed above.

Noting that, as the gap i increases, the bound for p decreases, we repeatedly apply Lemma 8, with increasing even gaps i, collecting at each iteration the finite list of prime pairs $(p, p+i)$ such that $\kappa(\{2, 3, 7, p, p+i\}) < 1/3$. For each i, the bound on p is found by solving the inequality

$$\frac{1}{3i} + \frac{2}{p} \leq \frac{2}{63}.$$

There are 17 pairs of primes $\boxed{(p, p+14)}$ such that $p < 252$:

$(3, 17)$ $(5, 19)$ $(17, 31)$ $(23, 37)$ $(29, 43)$ $(47, 61)$ $(53, 67)$

$(59, 73)$ $(83, 97)$ $(89, 103)$ $(113, 127)$ $(137, 151)$ $(149, 163)$ $(167, 181)$

$(179, 193)$ $(197, 211)$ $(227, 241)$.

Of these, only the set $\{2, 3, 7, 5, 19\}$ has kappa value less than 1/3.

There are 13 pairs of primes $\boxed{(p, p+16)}$ such that $p < 184$:

$(3, 19)$ $(7, 23)$ $(13, 29)$ $(31, 47)$ $(37, 53)$ $(43, 59)$ $(67, 83)$

$(73, 89)$ $(97, 113)$ $(151, 167)$ $(157, 173)$ $(163, 179)$ $(181, 197)$.

All of these have kappa value greater than 1/3.

There are 19 pairs of primes $\boxed{(p, p+18)}$ such that $p < 152$:

$(5, 23)$ $(11, 29)$ $(13, 31)$ $(19, 37)$ $(23, 41)$ $(29, 47)$ $(41, 59)$

$(43, 61)$ $(53, 71)$ $(61, 79)$ $(71, 89)$ $(79, 97)$ $(83, 101)$ $(89, 107)$

$(109, 127)$ $(113, 131)$ $(131, 149)$ $(139, 157)$ $(149, 167)$.

Of these, there are 3 such that the set $D = \{2, 3, 7, p, p+18\}$ has $\kappa(D) < 1/3$:

$(5, 23)$ $(19, 37)$ $(23, 41)$.

There are 12 pairs of primes $\boxed{(p, p+20)}$ such that $p < 133$:

$(3, 23)$ $(11, 31)$ $(17, 37)$ $(23, 43)$ $(41, 61)$ $(47, 67)$ $(53, 73)$

$(59, 79)$ $(83, 103)$ $(89, 109)$ $(107, 127)$ $(131, 151)$.

All of these have kappa value greater than 1/3.

There are 8 pairs of primes $\boxed{(p, p+22)}$ such that $p < 121$:

$(7, 29)$ $(19, 41)$ $(31, 53)$ $(37, 59)$ $(61, 83)$ $(67, 89)$ $(79, 101)$

$(109, 131)$.

Of these, there are 2 such that the set $D = \{2, 3, 7, p, p+22\}$ has $\kappa(D) < 1/3$:

$(19, 41)$ $(37, 59)$.

There are 17 pairs of primes $\boxed{(p, p+24)}$ such that $p < 112$:

$(5, 29)$ $(7, 31)$ $(13, 37)$ $(17, 41)$ $(19, 43)$ $(23, 47)$ $(29, 53)$

$(37, 61)$ $(43, 67)$ $(47, 71)$ $(59, 83)$ $(73, 97)$ $(79, 103)$ $(83, 107)$

$(89, 113)$ $(103, 127)$ $(107, 131)$.

Of these, there are 2 such that the set $D = \{2, 3, 7, p, p+24\}$ has $\kappa(D) < 1/3$:

$(5, 29)$ $(19, 43)$.

There are 10 pairs of primes $\boxed{(p, p+26)}$ such that $p < 106$:

$(3, 29)$ $(5, 31)$ $(11, 37)$ $(17, 43)$ $(41, 67)$ $(47, 73)$ $(53, 79)$

$(71, 97)$ $(83, 109)$ $(101, 127)$.

Of these, there are 2 such that the set $D = \{2, 3, 7, p, p+26\}$ has $\kappa(D) < 1/3$:

$(5, 31)$ $(11, 37)$.

There are 8 pairs of primes $\boxed{(p, p+28)}$ such that $p < 101$:

$(3, 31)$ $(13, 41)$ $(19, 47)$ $(31, 59)$ $(43, 71)$ $(61, 89)$ $(73, 101)$

$(79, 107)$.

Of these, only the set $\{2, 3, 7, 19, 47\}$ has kappa value less than $1/3$.

There are 17 pairs of primes $\boxed{(p, p+30)}$ such that $p < 97$:

$$(7, 37) \quad (11, 41) \quad (13, 43) \quad (17, 47) \quad (23, 53) \quad (29, 59) \quad (31, 61)$$
$$(37, 67) \quad (41, 71) \quad (43, 73) \quad (53, 83) \quad (59, 89) \quad (67, 97) \quad (71, 101)$$
$$(73, 103) \quad (79, 109) \quad (83, 113).$$

Of these, only the set $\{2, 3, 7, 11, 41\}$ has kappa value less than $1/3$.

There are 6 pairs of primes $\boxed{(p, p+32)}$ such that $p < 94$:

$$(5, 37) \quad (11, 43) \quad (29, 61) \quad (41, 73) \quad (47, 79) \quad (71, 103).$$

Of these, only the set $\{2, 3, 7, 5, 37\}$ has kappa value less than $1/3$.

There are 8 pairs of primes $\boxed{(p, p+34)}$ such that $p < 92$:

$$(3, 37) \quad (7, 41) \quad (13, 47) \quad (19, 53) \quad (37, 71) \quad (67, 101) \quad (73, 107)$$
$$(79, 113).$$

Of these, only the set $\{2, 3, 7, 19, 53\}$ has kappa value less than $1/3$.

There are 14 pairs of primes $\boxed{(p, p+36)}$ such that $p < 89$:

$$(5, 41) \quad (7, 43) \quad (11, 47) \quad (17, 53) \quad (23, 59) \quad (31, 67) \quad (37, 73)$$
$$(43, 79) \quad (47, 83) \quad (53, 89) \quad (61, 97) \quad (67, 103) \quad (71, 107) \quad (73, 109).$$

Of these, only the set $\{2, 3, 7, 5, 41\}$ has kappa value less than $1/3$.

There are 7 pairs of primes $\boxed{(p, p+38)}$ such that $p < 88$:

$$(3, 41) \quad (5, 43) \quad (23, 61) \quad (29, 67) \quad (41, 79) \quad (59, 97) \quad (71, 109).$$

Of these, only the set $\{2, 3, 7, 5, 43\}$ has kappa value less than 1/3.

There are 9 pairs of primes $\boxed{(p, p + 40)}$ such that $p < 86$:

$(3, 43)$ $(7, 47)$ $(13, 53)$ $(19, 59)$ $(31, 71)$ $(43, 83)$ $(61, 101)$

$(67, 107)$ $(73, 113)$.

All of these have kappa value greater than 1/3.

There are 13 pairs of primes $\boxed{(p, p + 42)}$ such that $p < 84$:

$(5, 47)$ $(11, 53)$ $(17, 59)$ $(19, 61)$ $(29, 71)$ $(31, 73)$ $(37, 79)$

$(41, 83)$ $(47, 89)$ $(59, 101)$ $(61, 103)$ $(67, 109)$ $(71, 113)$.

Of these, only the set $\{2, 3, 7, 5, 47\}$ has kappa value less than 1/3.

There are 6 pairs of primes $\boxed{(p, p + 44)}$ such that $p < 83$:

$(3, 47)$ $(17, 61)$ $(23, 67)$ $(29, 73)$ $(53, 97)$ $(59, 103)$.

Of these, only the set $\{2, 3, 7, 29, 73\}$ has kappa value less than 1/3.

There are 6 pairs of primes $\boxed{(p, p + 46)}$ such that $p < 82$:

$(7, 53)$ $(13, 59)$ $(37, 83)$ $(43, 89)$ $(61, 107)$ $(67, 113)$.

All of these have kappa value greater than 1/3.

There are 11 pairs of primes $\boxed{(p, p + 48)}$ such that $p < 81$:

$(5, 53)$ $(11, 59)$ $(13, 61)$ $(19, 67)$ $(23, 71)$ $(31, 79)$ $(41, 89)$

$(53, 101)$ $(59, 107)$ $(61, 109)$ $(79, 127)$.

Of these, there are 2 such that the set $D = \{2, 3, 7, p, p + 48\}$ has $\kappa(D) < 1/3$:

$$(5, 53) \qquad (19, 67).$$

There are 8 pairs of primes $\boxed{(p, p + 50)}$ such that $p < 80$:

$$(3, 53) \quad (11, 61) \quad (17, 67) \quad (23, 73) \quad (29, 79) \quad (47, 97) \quad (53, 103)$$
$$(59, 109).$$

All of these have kappa value greater than 1/3.

There are 5 pairs of primes $\boxed{(p, p + 52)}$ such that $p < 79$:

$$(7, 59) \quad (19, 71) \quad (31, 83) \quad (37, 89) \quad (61, 113).$$

All of these have kappa value greater than 1/3.

The following theorem summarizes this section. Note that the primes pairs $(p, p+i)$ such that $\kappa(\{2, 3, 7, p, p+i\}) < 1/3$ but $p < 79$ are not listed in this theorem. They will be covered in the next section.

Theorem 14. *If $i \geq 12$ and $p \geq 79$ and $D = \{2, 3, 7, p, p + i\}$ does not contain a proper subset that is class 4, then D is class 3 for any pair of primes $(p, p + i)$ not listed below:*

$$(89, 101) \quad (97, 109) \quad (139, 151) \quad (181, 193).$$

4.2 Applying Lemma 7

The next step in the process is to remove the bound that p must be greater than 79. This is accomplished applying Lemma 7 to each set $\{2, 3, 7, p\}$ for primes

$p < 79$. The fact that we switch from using Lemma 8 to Lemma 7 at $i = 52$ and $p < 79$ is arbitrary. Computational, the hardest part of using Lemma 7 is finding the length of the longest interval in $\{2, 3, 7, p\}$, which is why Lemma 8 was used as long as it was. Now, for each prime $11 \leq p < 79$, a bound on q such that $\{2, 3, 7, p, q\}$ is class 3 is established. To finish each case we check the finite list of small primes q.

For some of the smallest primes p, the prime sets $D = \{2, 3, 7, p, q\}$ are class 3 for all primes q such that D does not contain a proper subset known to be class 4.

Theorem 15. *The set $\{2, 3, 7, 11, p\}$ is class 3 for all primes $p \notin \{5, 13, 19, 23, 37, 41\}$.*

Proof. Apply Lemma 7 to the set $D = \{2, 3, 7, 11\}$. Since $I_D(1/3) = [7/33, 2/9] \cup [7/9, 26/33]$, a longest connected subset has length $1/99$. First we solve the inequality

$$p \geq \frac{\frac{2}{3}}{\frac{1}{99}}$$

for p. Thus, if $p \geq 66$, then by Lemma 7 $\{2, 3, 7, 11, p\}$ is class 3. Calculating the kappa values for the sets $\{2, 3, 7, 11, p\}$ for all primes $5 \leq p < 66$ shows that only those sets listed in the statement have kappa value less than $1/3$.

□

Theorem 16. *The set $\{2, 3, 7, 13, p\}$ is class 3 for all $p \notin \{5, 11\}$.*

Proof. Similar to Theorem 15, we start by applying Lemma 7. As $I_{\{2,3,7,13\}} \supset [4/21, 8/39]$, we can calculate that if $p \geq 46$, then $\{2, 3, 7, 13, p\}$ is class 3. Again, calculations show that only $\{2, 3, 7, 13, 5\}$ and $\{2, 3, 7, 13, 11\}$ have kappa value less than $1/3$. □

Theorem 17. *The set $\{2, 3, 7, 17, p\}$ is class 3 for all $p \notin \{5, 19, 29\}$.*

Proof. Noting that $[10/51, 11/51] \subset I_{\{2,3,7,17\}}$, by Lemma 7 we calculate that if $p \geq 34$,

Table 4.1: *Applying Lemma 7 to* $\{2, 3, 7, p\}$ *for primes* $23 \leq p < 79$

p	$[a, b] \subset I_{\{2,3,7,p\}}$	Bound on q	Primes q with $\kappa(\{2, 3, 7, p, q\}) < 1/3$
23	$[4/21, 14/69]$	54	5,11,31,41
29	$[4/21, 17/87]$	136	5,17,31,37,41,73,109
31	$[19/93, 20/93]$	62	5,19,23,29,43
37	$[22/111, 23/111]$	74	5,11,19,29,59
41	$[25/123, 26/123]$	82	5,11,19,23,29,43,53
43	$[25/129, 26/129]$	86	5,19,31,41
47	$[28/141, 29/141]$	94	5,19,59
53	$[34/159, 35/159]$	106	5,19,41
59	$[37/177, 38/177]$	118	5,37,47,61
61	$[37/183, 38/183]$	122	5,59,73
67	$[43/201, 44/201]$	134	5,19,79
71	$[46/213, 47/213]$	142	5,73,83
73	$[46/219, 47/219]$	146	5,19,29,61,71

then $\{2, 3, 7, 17, p\}$ is class 3. Calculations show that only those sets listed in the statement have kappa value less than 1/3. □

We now consider the first prime for which we uncover prime sets $D = \{2, 3, 7, p, q\}$ for which $\kappa(D) < 1/3$ and no subset of D is class 4.

Theorem 18. *The set* $\{2, 3, 7, 19, p\}$ *is class 3 for all primes*

$$p \notin \{5, 11, 17, 31, 37, 41, 43, 47, 53, 67, 73, 79, 83, 89, 109, 131, 151, 157, 167, 193\}.$$

Proof. If $p \geq 266$, then $\{2, 3, 7, 19, p\}$ is class 3 by Lemma 7 since $[4/21, 11/57] \subseteq I_{\{2,3,7,19\}}$. Computer calculations confirm that $\kappa(\{2, 3, 7, 19, p\}) \geq 1/3$ if $p < 266$ and p is not an element of the set in the statement. □

For the rest of the primes less than 79, see Table 4.1 for a summary of the bounds on q and the sets for each p that have kappa value less than 1/3. The following theorem summarizes this section.

Theorem 19. *If* $p \leq 79$*,* $q > p$ *and* $D = \{2, 3, 7, p, q\}$ *does not contain any proper* *subset that is class 4, then D is class 3 for any pair of primes* (p, q) *not listed below:*

$(19, 31)$	$(19, 37)$	$(19, 41)$	$(19, 43)$	$(19, 47)$	$(19, 53)$	$(19, 67)$
$(19, 73)$	$(19, 79)$	$(19, 83)$	$(19, 89)$	$(19, 109)$	$(19, 131)$	$(19, 151)$
$(19, 157)$	$(19, 167)$	$(19, 193)$	$(29, 41)$	$(29, 73)$	$(29, 109)$	$(31, 43)$
$(37, 59)$	$(41, 53)$	$(47, 59)$	$(61, 73)$	$(67, 79)$	$(71, 83)$.	

4.3 Applying Lemma 9

Thus far we have shown that, as long as $i \geq 12$, there are only finitely many prime sets with $\kappa(\{2, 3, 7, p, p+i\}) < 1/3$. If $i = 2$, then p and $p+2$ are twins primes and the set is class 4. The last step in the process is to show that, for $i \in \{4, 6, 8, 10\}$, all prime sets of the form $\{2, 3, 7, p, p+i\}$ that do not contain one of the known class 4 sets are class 3.

Consider the case when p and $p+4$ are both primes. Note that this implies that $p \equiv 1 \pmod 6$. We want to apply Lemma 9 to check if any rational points in the interval $[4/21, 2/9] \subset I_{\{2,3,7\}}$ are in both I_p and I_{p+4}. A natural place to start is by taking the least common multiple of 6, 21 and 9, which is 126. The target interval $[4/21, 2/9] = [24/126, 28/126]$, so we will apply Lemma 9 for the points $\{n/126 : 24 \leq n \leq 28\}$. After removing the residue classes modulo 126 for which $p \not\equiv 1 \pmod 6$, we are left with the Table 4.2.

From Table 4.2 we see that, for each of the rows that is not highlighted, $I_{\{2,3,7,p,p+4\}}$ will contain the point in the rightmost column, implying that $\{2, 3, 7, p, p+4\}$ is class 3. To investigate further, we increase the number of rational points to check

Table 4.2: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+4\}}$ *(Round 1)*

$p \pmod{126}$	$\gcd(p, 126)$	$\gcd(p+4, 126)$	Point in $I_{\{p,p+4\}}$
1			
7	7		3/14
13			25/126
19			4/21
25			2/9
31		7	3/14
37			13/63
43			2/9
49	7		3/14
55			
61			4/21
67			
73		7	3/14
79			2/9
85			13/63
91	7		3/14
97			2/9
103			4/21
109			25/126
115		7	3/14
121			

by a factor of 5. We must also expand our list of residues to check, so we get Table 4.3.

From Table 4.3 we see that if $p \equiv 1 \pmod{630}$, then $p+4$ is not prime, if $p \equiv 625 \pmod{630}$, then p is not prime, and if $p \not\equiv 307, 319 \pmod{630}$, then $I_{\{2,3,7,p,p+4\}}$ is not empty. Iterating again, this time just increasing by a factor of 2 gives Table 4.4, which has no highlighted rows. This means, no matter the residue class of a prime p modulo 1260, there exists some point in $I_{\{2,3,7,p,p+i\}}$. Thus, this is the final table needed to finish the case when $i = 4$. Tables 4.2 to 4.4 show that $\{2, 3, 7, p, p+4\}$ is class 3 for every pair of primes p and $p+4$.

Next, let $i = 6$. We begin in the same way as for $i = 4$, by checking the

Table 4.3: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+4\}}$ *(Round 2)*

$p \pmod{630}$	$\gcd(p, 630)$	$\gcd(p+4, 630)$	Point in $I_{\{p,p+4\}}$
1		5	
55	5		61/315
67			64/315
121		5	41/210
127			61/315
181		5	62/315
193			1/5
247			62/315
253			121/630
307			
319			
373			121/630
379			62/315
433			1/5
445	5		62/315
499			61/315
505	5		41/210
559			64/315
571		5	61/315
625	5		

Table 4.4: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+4\}}$ *(Round 3)*

$p \pmod{1260}$	$\gcd(p, 1260)$	$\gcd(p+4, 1260)$	Point in $I_{\{p,p+4\}}$
307			253/1260
319			251/1260
937			251/1260
949			253/1260

Table 4.5: *Rational points in $I_{\{2,3,7\}} \cap I_{\{p,p+6\}}$ (Round 1)*

$p \pmod{126}$	$\gcd(p, 126)$	$\gcd(p+6, 126)$	Point in $I_{\{p,p+6\}}$
1		7	
5			
7	7		4/21
11			3/14
13			4/21
17			25/126
19			
23			4/21
25			3/14
29		7	4/21
31			13/63
35	7		
37			25/126
41			
43		7	
47			
49	7		4/21
53			3/14
55			4/21
59			
61			
65			4/21
67			3/14
71		7	4/21
73			
77	7		
79			
83			25/126
85		7	
89			13/63
91	7		4/21
95			3/14
97			4/21
101			
103			25/126
107			4/21
109			3/14
113		7	4/21
115			
119	7		
121			
125			

Table 4.6: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+6\}}$ *(Round 2)*

$p \pmod{630}$	$\gcd(p, 630)$	$\gcd(p+6, 630)$	Point in $I_{\{p,p+6\}}$
⋮			⋮
293			22/105
299		5	121/630
311			
313			
325	5		121/630
331			22/105
⋮			⋮

Table 4.7: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+6\}}$ *(Round 3)*

$p \pmod{1260}$	$\gcd(p, 1260)$	$\gcd(p+6, 1260)$	Point in $I_{\{p,p+6\}}$
311			241/1260
313			27/140
941			27/140
943			241/1260

points $\{n/126 : 24 \leq n \leq 28\}$, noting that p can be either 1 or 5 $\pmod 6$. This gives Table 4.5. We iterate the process, increasing the number of points checked by a factor of 5. This gives Table A.1, which is too long to fit on a single page, so it is moved to an appendix. The only highlighted rows are shown in Table 4.6. Finally, we iterate again, increasing by a factor of 2. Table 4.7 shows that this finishes the case when $i = 6$.

The case when $i = 8$ is much more difficult, and this is not surprising, as we have already have seen that

$$\{2, 3, 5, 13\} \quad \{2, 3, 11, 19\} \quad \{2, 3, 23, 31\} \quad \{2, 3, 29, 37\}$$

are all class 4 sets. Using similar methods to those above, we were able to show that, if $\{2, 3, 7, p, p + 8\}$ is class 4, then $p \equiv 2311139, 2311163 \pmod{4622310}$. Note that $4622310 = 2 \cdot 3^2 \cdot 5 \cdot 7 \cdot 11 \cdot 23 \cdot 29$. See Appendix B.

Finally, consider the case when $i = 10$. We can again assume that $p \equiv 1 \pmod 6$, so we start with Table 4.8, checking points $\{n/126 : 24 \leq n \leq 28\}$. From here increase the number of rational points checked by a factor of 5 to get Table 4.9. And finally, increase by a factor of 11 to get Table 4.10. Notice that by checking points such that 11 divides the denominator allows us to remove the possibility that $p \equiv 1 \pmod{6930}$, since this would imply that $p + 10$ is not prime.

Table 4.8: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+10\}}$ *(Round 1)*

$p \pmod{126}$	$\gcd(p, 126)$	$\gcd(p+10, 126)$	Point in $I_{\{p,p+10\}}$
1			
7	7		25/126
13			4/21
19			4/21
25		7	3/14
31			13/63
37			
43			25/126
49	7		3/14
55			4/21
61			4/21
67		7	3/14
73			25/126
79			
85			13/63
91	7		3/14
97			4/21
103			4/21
109		7	25/126
115			
121			

Table 4.9: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+10\}}$ *(Round 2)*

$p \pmod{630}$	$\gcd(p, 630)$	$\gcd(p+10, 630)$	Point in $I_{\{p,p+10\}}$
1			
37			1/5
79			41/210
115	5	5	41/210
121			61/315
127			61/315
163			61/315
205	5	5	
241			62/315
247			121/630
253			121/630
289			121/630
331			121/630
367			121/630
373			121/630
379			62/315
415	5	5	
457			61/315
493			61/315
499			61/315
505	5	5	41/210
541			41/210
583			1/5
619			
625	5	5	

Table 4.10: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+10\}}$ *(Round 3)*

$p \pmod{6930}$	$\gcd(p, 6930)$	$\gcd(p+10, 6930)$	Point in $I_{\{p,p+10\}}$
1		11	
619			221/1155
631			21/110
1249			21/110
1261			661/3465
1879			661/3465
1891			1321/6930
2509		11	661/3465
2521			1321/6930
3139			221/1155
3151			1321/6930
3769			1321/6930
3781			221/1155
4399			1321/6930
4411	11		661/3465
5029			1321/6930
5041			661/3465
5659			661/3465
5671			21/110
6289			21/110
6301			221/1155
6919	11		

The following is a summary of this chapter.

Theorem 20. *A prime set of the form $D = \{2, 3, 7, p, q\}$ is class 3 if none of the following is true:*

(1) *D contains a proper subset that is class 4.*

(2) *The pair (p, q) is one of the following 31 pairs:*

$(19, 31)$	$(19, 37)$	$(19, 41)$	$(19, 43)$	$(19, 47)$	$(19, 53)$	$(19, 67)$
$(19, 73)$	$(19, 79)$	$(19, 83)$	$(19, 89)$	$(19, 109)$	$(19, 131)$	$(19, 151)$
$(19, 157)$	$(19, 167)$	$(19, 193)$	$(29, 41)$	$(29, 73)$	$(29, 109)$	$(31, 43)$
$(37, 59)$	$(41, 53)$	$(47, 59)$	$(61, 73)$	$(67, 79)$	$(71, 83)$	$(89, 101)$
$(97, 109)$	$(139, 151)$	$(181, 193)$.				

(3) $p \equiv 122491199, 122491223 \pmod{244982430}$ *and* $q = p + 8$.

CHAPTER 5

Class 4 Prime Sets of the Form $\{2, 3, 7, 19, p\}$

The kappa value can be used to prove that a set is class 3, but in order to establish that a set is class 4 we need other tools. In the following section we investigate 3-colorings of the distance graph generated by $\{2, 3, 7, 19\}$ and show that these colorings cannot be extended to $\{2, 3, 7, 19, p\}$ for certain p.

5.1 Background

In this section, our notation will follow that of Eggleton in [11]. Given a set D of positive integers, a *D-consistent 3-coloring* is a function $c\colon \boldsymbol{Z} \to \{0, 1, 2\}$ such that for every $i, j \in \boldsymbol{Z}$,

$$|i - j| \in D \implies c(i) \neq c(j).$$

In the following we will consider a coloring c as a two-way infinite sequence, $\boldsymbol{c} := \{c(i)\}_{i\in\boldsymbol{Z}}$.

The structure of a coloring sequence $\boldsymbol{c}$ can be described by breaking it apart into the three constituent color classes. The *k-color-class* is defined as the set $\{i \in \boldsymbol{Z} : c(i) = k\}$. Since each block of five consecutive integers in the distance graph generated by $\{2, 3\}$ contains the 5-cycle $\{i+1, i+3, i+5, i+2, i+4\}$, the difference between any two consecutive elements in a color class is at most 5, otherwise the five cycle must be properly colored with just two colors, which is impossible. In light of this we can consider each color class as a strictly increasing sequence of integers $\boldsymbol{k} := \{k_i\}_{i\in\boldsymbol{Z}}$ where $c(k_i) = k$ for every i and $k_i < k_{i+1}$. The structure of a color class is primarily captured by the gaps or differences between consecutive elements in the ordered color class sequence. The *gap sequence* of a k-color-class $\boldsymbol{k}$ is defined as $\Delta_k(\boldsymbol{c}) = \{d_i\}_{i \in \boldsymbol{Z}}$ where $d_i = k_{i+1} - k_i$.

For either a color sequence or a gap sequence, we call any finite set of consecutive terms a *block* of the sequence. For any gap sequence $\boldsymbol{d}$, let $\sigma(\boldsymbol{d})$ be the set of all partial sums obtained by summing the terms in all blocks of $\boldsymbol{d}$. Given a coloring $\boldsymbol{c}$, let $\sigma(\boldsymbol{c}) := \sigma(\Delta_0(\boldsymbol{c})) \cup \sigma(\Delta_1(\boldsymbol{c})) \cup \sigma(\Delta_2(\boldsymbol{c}))$. The following proposition from [11] connects the representation of 3-colorings as gap sequence triples to the D-consistency of the coloring.

Proposition 21. *For any coloring* $\mathbf{c}$ *and any fixed* $a \in \boldsymbol{Z}^+$*, there exists an* $i \in \boldsymbol{Z}$ *such that* $c(i) = c(i+a)$ *if and only if* $a \in \sigma(\boldsymbol{c})$.

Often the colorings considered are periodic. This is denoted by enclosing the repeated block in parenthesis. As an example of these definitions, consider the periodic coloring function c defined by

$$c(i) = \begin{cases} 0 \text{ if } i \equiv 0, 1, 5, 6, 10, 11, 16 \pmod{21} \\ 1 \text{ if } i \equiv 2, 7, 8, 12, 13, 17, 18 \pmod{21} \\ 2 \text{ if } i \equiv 3, 4, 9, 14, 15, 19, 20 \pmod{21}. \end{cases}$$

The corresponding coloring sequence is $\boldsymbol{c} = (001220011200112201122)$, and the three color classes are:

$$\mathbf{0} = \{\ldots 0, 1, 5, 6, 10, 11, 16, \ldots\}$$

$$\mathbf{1} = \{\ldots 2, 7, 8, 12, 13, 17, 18, \ldots\}$$

$$\mathbf{2} = \{\ldots 3, 4, 9, 14, 15, 19, 20, \ldots\}.$$

The three gap sequences are:

$$\Delta_0(c) = (1, 4, 1, 4, 1, 5, 5)$$

$$\Delta_1(c) = (5, 1, 4, 1, 4, 1, 5)$$

$$\Delta_2(c) = (1, 5, 5, 1, 4, 1, 4).$$

Since each of these gap sequences is a cyclic permutation of the others, the partial sums are the same for each:

$$\sigma(\Delta_0(\boldsymbol{c})) = \sigma(\boldsymbol{c}) = \{x : x \equiv 0, \pm 1, \pm 4, \pm 5, \pm 6, \pm 9, \pm 10 \pmod{21}\}$$

Thus, since the intersection of $\{2, 3, 7, 19\}$ and $\sigma(\boldsymbol{c})$ is empty, by Proposition 21, $\boldsymbol{c}$ is a $\{2, 3, 7, 19\}$-consistent 3-coloring.

5.2 Characterizing gap sequences

In this section we will investigate what blocks are possible for the gap sequences of a {2,3,7,19}-consistent coloring. Blocks of length l will be called l-blocks. In order to show that certain blocks are not possible, we will need to investigate how all three color classes interact. A gap sequence $\boldsymbol{d}$ almost completely determines a color sequence, as made precise by the following proposition from [11]:

Proposition 22. *If $\boldsymbol{d}$ is a $\{2, 3\}$-consistent gap sequence, then $\boldsymbol{d} = \Delta_0(\boldsymbol{c})$ where, up to a permutation of the labels, $\mathbf{c}$ is given by the following rule that assigns terms of the gap sequence to blocks of a color sequence:*

$$\theta(d_i) = \begin{cases} 0 & \textit{if } d_i = 1 \\ 0112 & \textit{if } d_{i-1} > 1 \textit{ and } d_i = 4 \textit{ and } d_{i+1} = 1 \\ 01z2 & \textit{if } d_{i-1} = 1 \textit{ and } d_i = 4 \textit{ and } d_{i+1} = 1 \\ 0122 & \textit{if } d_{i-1} = 1 \textit{ and } d_i = 4 \textit{ and } d_{i+1} > 1 \\ 01122 & \textit{if } d_i = 5 \end{cases}$$

where $z \in \{1, 2\}$ *can be arbitrarily chosen for each* 141 *block in* $\boldsymbol{d}$.

The only possible gaps between consecutive elements of a color class are 1,4 and 5. The fact that 2 or 3 cannot be gaps follows clearly from the definition, and the fact that no gap can be greater than 5 follows from existence of a 5-cycle in any block of five consecutive integers.

There are 9 possible 2-blocks of 1,4, and 5: 11, 14, 15, 41, 44, 45, 51, 54, 55. Of these, 11, 44, 45, 54 cannot be 2-blocks of a $\{2, 3, 7, 19\}$-consistent gap sequence. The fact that 11 is impossible follows clearly from the fact that it contains a partial sum of 2.

Proposition 23. *Any* $\{2, 3, 7, 19\}$*-consistent gap sequence cannot contain the 2-block* 44.

Proof. Let $\boldsymbol{d}$ be a $\{2, 3, 7, 19\}$-consistent gap sequence containing a 44 block. By Proposition 22, the corresponding color sequence must have the form $\boldsymbol{c} = \ldots 012201120 \ldots$. Without loss of generality, let $c_0 = 0$, $c_1 = 1$, $c_3 = 2$, etc. We can now make the following chain of inferences:

$$(c_7 = 2) \wedge (c_8 = 0) \implies c_{10} = 1$$

$$(c_0 = 0) \wedge (c_1 = 1) \implies c_{-2} = 2$$

$$(c_{-2} = 2) \wedge (c_{10} = 1) \implies c_{17} = 0$$

$$(c_7 = 2) \wedge (c_{17} = 0) \implies c_{14} = 1$$

$$(c_8 = 0) \wedge (c_{14} = 1) \implies c_{11} = 2$$

$$(c_6 = 1) \wedge (c_7 = 2) \implies c_9 = 0$$

$$(c_7 = 2) \wedge (c_8 = 0) \implies c_{10} = 1$$

$$(c_9 = 0) \wedge (c_{10} = 1) \implies c_{12} = 2$$

$$(c_{10} = 1) \wedge (c_{11} = 2) \implies c_{13} = 0$$

$$(c_{12} = 2) \wedge (c_{13} = 0) \implies c_{15} = 1$$

$$(c_{13} = 0) \wedge (c_{14} = 1) \implies c_{16} = 2$$

$$(c_{15} = 1) \wedge (c_{16} = 2) \implies c_{18} = 0.$$

The fact that $c_2 = 2$, $c_{14} = 1$ and $c_{18} = 0$ implies that c_{21} cannot be properly colored, contradicting that $\boldsymbol{d}$ is a $\{2, 3, 7, 19\}$-consistent gap sequence. □

Proposition 24. *Any* $\{2, 3, 7, 19\}$*-consistent gap sequence cannot contain the 2-block 45.*

Proof. Let $\boldsymbol{d}$ be a $\{2, 3, 7, 19\}$-consistent gap sequence containing the 2-block 45. By Proposition 22, we can assume the associated coloring sequence $\boldsymbol{c}$ contains the following block: $c_0 \dots c_9 = 0122011220$. Then

$$(c_4 = 0) \wedge (c_8 = 2) \implies c_{11} = 1$$

$$(c_5 = 1) \wedge (c_9 = 0) \implies c_{12} = 2$$

$$(c_0 = 0) \wedge (c_{12} = 2) \implies c_{19} = 1$$

$$(c_9 = 0) \wedge (c_{19} = 1) \implies c_{16} = 2$$

$$(c_{11} = 1) \wedge (c_{12} = 2) \implies c_{14} = 0$$

$$(c_{11} = 1) \wedge (c_{16} = 2) \implies c_{13} = 0$$

$$(c_{12} = 2) \wedge (c_{13} = 0) \implies c_{15} = 1$$

$$(c_{14} = 0) \wedge (c_{15} = 1) \implies c_{17} = 2$$

$$(c_{15} = 1) \wedge (c_{16} = 2) \implies c_{18} = 0$$

The fact that $c_1 = 1$, $c_{17} = 2$ and $c_{18} = 0$ implies that c_{20} cannot be properly colored. □

Proposition 25. *Any* $\{2, 3, 7, 19\}$*-consistent gap sequence cannot contain the 2-block* 54*.*

Proof. Let $\boldsymbol{d}$ be a $\{2, 3, 7, 19\}$-consistent gap sequence containing the 2-block 54. By Proposition 22, we can assume the associated coloring sequence $\boldsymbol{c}$ contains the following block: $c_0 \ldots c_9 = 0112201120$. Then

$$(c_7 = 1) \wedge (c_8 = 2) \implies c_{10} = 0$$

$$(c_8 = 2) \wedge (c_9 = 0) \implies c_{11} = 1$$

$$(c_{10} = 0) \wedge (c_{11} = 1) \implies c_{13} = 2$$

$$(c_9 = 0) \wedge (c_{13} = 2) \implies c_{16} = 1$$

$$(c_1 = 1) \wedge (c_{13} = 2) \implies c_{20} = 0$$

The fact that $c_4 = 2$, $c_{16} = 1$ and $c_{20} = 0$ implies that c_{23} cannot be properly colored. □

From the five allowable 2-blocks, 9 3-blocks can be built: 141, 151, 155, 414, 415, 514, 515, 551, 555. Of these, both 515 and 151 are not possible blocks of a $\{2, 3, 7, 19\}$-consistent gap sequence. 151 produces a partial sum of 7, and is therefore not possible.

Proposition 26. *Any* $\{2, 3, 7, 19\}$*-consistent gap sequence cannot contain the 3-block* 515*.*

Proof. Let $\boldsymbol{d}$ be a $\{2, 3, 7, 19\}$-consistent gap sequence containing the 3-block 515. By Proposition 22, we can assume the associated coloring sequence $\boldsymbol{c}$ contains the following block: $c_0 \ldots c_{11} = 011220011220$. Then the fact that $c_1 = 1$ and $c_8 = 1$ contradicts the fact that $\boldsymbol{c}$ is a proper coloring. □

Finally three larger blocks are not allowed: 5555, 14141414 and 51415. The block 14141414 contains a partial sum of 19, and therefore cannot be in a $\{2, 3, 7, 19\}$-consistent gap sequence.

Proposition 27. *Any* $\{2, 3, 7, 19\}$*-consistent gap sequence cannot contain the block*

5555.

Proof. Assume $\boldsymbol{d}$ is a $\{2,3,7,19\}$-consistent gap sequence containing 5555. By Proposition 22, the associated color sequence contains the following block:

$$c_0 \ldots c_{19} = 01122011220112201122.$$

The fact that $c_1 = 1$ and $c_{13} = 2$ implies $c_{20} = 0$, but this together with the fact that $c_4 = 2$ and $c_{16} = 1$ means that c_{23} cannot be properly colored. □

Proposition 28. *Any* $\{2,3,7,19\}$*-consistent gap sequence cannot contain the block* 51415.

Proof. Assume $\boldsymbol{d}$ is a $\{2,3,7,19\}$-consistent gap sequence containing the block 51415. By Proposition 22, the associated color sequence must contain the following block:

$$c_0 \ldots c_{15} = 01122001x2001122$$

where $c_8 = x$ is not determined by the θ-rule. But the fact that $c_1 = 1$, $c_6 = 0$ and $c_{15} = 2$ implies that c_8 cannot be properly colored. □

With the above classification of allowable blocks, we can characterize the possible $\{2,3,7,19\}$-consistent 3 colorings. The fact that 151, 45, 54 and 5555 are all impossible implies that any time a 5 occurs it must be part of a 1551 or a 15551 block. The fact that 11, 44 and 14141414 are all impossible implies that a 5 must occur in all gap sequences. The fact that 515 and 51415 are impossible implies that every gap sequence has the following form:

$$\boldsymbol{d} = \sum_{i \in \boldsymbol{Z}} A_i B_i$$

where $A_i \in \{1414, 141414\}$ and $B_i \in \{155, 1555\}$, and the summation is representing concatenation of blocks.

Thus any $\{2, 3, 7, 19\}$-consistent gap sequence is built from the following four blocks:

$$C_1 = 1414155, \qquad C_2 = 14141555, \qquad C_3 = 141414155, \qquad C_4 = 1414141555.$$

5.3 Characterizing color sequences

The monochromatic gap sequences are not sufficient to classify all sets $\{2, 3, 7, 19, p\}$, as $43 \notin \sigma(\boldsymbol{d})$ when $\boldsymbol{d} := (C_1C_2)$. We must consider the full color sequences. As we are concerned with $\{2, 3, 7, 19\}$-consistent colorings we can strengthen Proposition 22 in the following way:

Lemma 29. *If* $\boldsymbol{d}$ *is a* $\{2, 3, 7, 19\}$*-consistent gap sequence, then* $\boldsymbol{d} = \Delta_0(\boldsymbol{c})$ *where, up to a permutation of the labels,* $\mathbf{c}$ *is given by the following rule:*

$$\eta(d_i) = \begin{cases} 0 & \text{if } d_i = 1 \\ 0112 & \text{if } d_{i-6} \cdots d_i = 5551414 \text{ or } d_i \cdots d_{i+2} = 415 \\ 01z2 & \text{if } d_{i-6} \cdots d_{i+6} = 1551414141551 \\ 0122 & \text{if } d_{i-2} \cdots d_i = 514 \text{ or } d_i \cdots d_{i+6} = 4141555 \\ 01122 & \text{if } d_i = 5 \end{cases}$$

where $z \in \{1, 2\}$ *can be chosen arbitrarily for each* 1551414141551 *block in* $\boldsymbol{d}$*.*

Proof. By Proposition 22, we need only prove the cases where $d_i = 4$.

Case 1: Suppose $d_{i-6} \cdots d_i = 5551414$. Then

$$\theta(d_{i-6} \cdots d_i) = 01122011220112200 1z_1 2001z_2 2.$$

The integer 19 spaces before z_2 is colored with a 2, so $\eta(d_i) = 0112$.

Case 2: Suppose $d_i d_{i+1} d_{i+2} = 415$. Then

$$\theta(d_i d_{i+1} d_{i+2}) = 01z2001122.$$

The integer 7 spaces after z is colored with a 2, so $\eta(d_i) = 0112$.

Case 3: Suppose $d_{i-2} d_{i-1} d_i = 514$. Then

$$\theta(d_{i-2} d_{i-1} d_1) = 01122001z2.$$

The integer 7 spaces before z is colored with a 1, so $\eta(d_i) = 0122$.

Case 4: Suppose $d_i \cdots d_{i+6} = 4141555$. Then

$$\theta(d_i \cdots d_{i+6}) = 01z_1 2001z_2 200112201122 01122.$$

The integer 19 spaces after z_1 is colored with a 1, so $\eta(d_i) = 0122$.

Case 5: If d_i does not fall under cases 2 or 3, then it must be in a block of the form:

$$d_{i-5} \cdots d_{i+5} = 55141414155.$$

If either d_{i-6} or d_{i+6} is 5, then it falls under either case 1 or 4. Note that it cannot be both, since then the indeterminate color z in $\theta(d_i)$ cannot be properly colored. Thus the only block not cover by the previous cases is the following:

$$d_{i-6} \cdots d_{i+6} = 1551414141551,$$

where the indeterminate color z in $\theta(d_i)$ can still be either 1 or 2. □

Our four gap sequence blocks can now be expanded to color sequence blocks. The strengthened η completely determines the color sequences from C_1, C_2 and C_4. The block C_3 can expand into two different color sequence blocks, depending on the choice for z.

$$A_1 := \eta(C_1) = 00122001120011220 1122$$

$$A_2 := \eta(C_2) = 00122001120011220112201122$$

$$A_3 := \eta(C_3) = 00122001120011200112201122 \quad \text{(with } z = 1\text{)}$$

$$A_3' := \eta(C_3) = 00122001220011200112201122 \quad \text{(with } z = 2\text{)}$$

$$A_4 := \eta(C_4) = 0012200122001120011220112201122.$$

It is more convenient to work with gap sequence triples rather than undifferentiated color sequences, so we unravel the above color sequences into the gap sequences for each color class. Note that, in order to get the last number for the gap sequences, the fact that each of the color sequences above start with the block 0012 is used.

$$\Delta_0(A_1) = 1414155 \qquad \Delta_1(A_1) = 5141415 \qquad \Delta_2(A_1) = 1551414$$

$$\Delta_0(A_2) = 14141555 \qquad \Delta_1(A_2) = 514141415 \qquad \Delta_2(A_2) = 155141414$$

$$\Delta_0(A_3) = 141414155 \qquad \Delta_1(A_3) = 514141415 \qquad \Delta_2(A_3) = 15551414$$

$$\Delta_0(A_3') = 141414155 \qquad \Delta_1(A_3') = 55141415 \qquad \Delta_2(A_3') = 141551414$$

$$\Delta_0(A_4) = 1414141555 \qquad \Delta_1(A_4) = 5514141415 \qquad \Delta_2(A_4) = 14155141414.$$

Thus any color sequence $\boldsymbol{c}$ must have the form:

$$\boldsymbol{c} = \sum_{i \in \boldsymbol{Z}} X_i,$$

where $X_i \in \{A_1, A_2, A_3, A_3', A_4\}$. But we need to put some restrictions on which blocks can follow one another. From Δ_2, it is clear that A_4 cannot be followed by either A_3' or A_4, since this would create a 14141414 block. Similarly A_2 cannot be followed by either A_3' or A_4. Otherwise the blocks can be freely concatenated.

5.4 Guaranteed partial sums

Recall that computer calculations show that if

$$p \in X := \{31, 37, 41, 43, 47, 53, 67, 73, 79, 83, 89, 109, 131, 151, 157, 167, 193\},$$

then $\kappa(\{2, 3, 7, 19, p\}) < 1/3$. In this section we will show that no $\{2, 3, 7, 19\}$-consistent coloring can be extended to a $\{2, 3, 7, 19, p\}$-consistent coloring for any $p \in X$. This suffices to classify $\{2, 3, 7, 19, p\}$ as class 4.

Theorem 30. *If $p \in \{31, 37, 41\}$, then $\{2, 3, 7, 19, p\}$ is class 4.*

Proof. Let $p \in \{31, 37, 41\}$, and assume that $\boldsymbol{c}$ is a $\{2, 3, 7, 19, p\}$-consistent 3-coloring. We know that $\boldsymbol{d} := \Delta_0(\boldsymbol{c})$ must contain at least one of the blocks C_1, C_2, C_3 or C_4. Let $|C_i|$ denote the sum of all the terms in C_i. Then $|C_1| = 21$, $|C_2| = |C_3| = 26$, and $|C_4| = 31$. By the structure of $\{2, 3, 7, 19\}$-consistent gap sequences, we know that, regardless of what block precedes or follows C_i, the sequence must have the form

$$\boldsymbol{d} = \cdots 55C_i14141 \cdots$$

Thus we know $\sigma(\boldsymbol{d})$ will contain the set $\{|C_i| + n : n \in \{1, 5, 6, 10, 11, 15, 16, 20, 21\}\}$.

Since

$$31 = |C_1| + 10 = |C_2| + 5 \; = |C_3| + 5 \; = |C_4|,$$

$$37 = |C_1| + 16 = |C_2| + 11 = |C_3| + 11 = |C_4| + 6,$$

$$41 = |C_1| + 20 = |C_2| + 15 = |C_3| + 15 = |C_4| + 10,$$

we know that $\{21, 37, 41\} \subset \sigma(\boldsymbol{d})$, and by Proposition 21 this contradicts the claim that $\boldsymbol{c}$ is a $\{2, 3, 7, 19, p\}$-consistent 3-coloring. □

Theorem 31. $\{2, 3, 7, 19, 43\}$ *is class 4.*

Proof. Assume that $\boldsymbol{c}$ is a $\{2, 3, 7, 19, 43\}$-consistent 3-coloring.

Case 1: Let $\boldsymbol{c}$ contain A_1. Since $|\Delta_k(A_1)| = 21$ for each k, we must show that we can always add 22 to the end of a block. Each Δ_2 has an initial sum of 22, noting that $\Delta_2(A_1)$, which has length 21, is always followed by another 1.

Case 2: Let $\boldsymbol{c}$ contain A_2, A_3 or A_3'. Each of these blocks have length of 26. Thus we must show that we can add 17. Again, each Δ_2 has an initial sum of 17.

Case 3: Let $\boldsymbol{c}$ contain A_4. Since $|A_4| = 31$, we must show that we can add 12. Since the block after A_4 cannot be A_3' or A_4, $\Delta_1(\boldsymbol{c})$ must have the form:

$$\cdots 15\Delta_1(A_4)51 \cdots$$

Adding both sides gives 12, as required. □

For the rest of the primes, the arguments only get more involved. We leave the verification that the partial sums of each color sequence of the prescribed form contains each $p \in X$ to a computer (see Appendix C). To do so we construct an infinite tree `colorings` shown in Fig. 5.1. The tree is mutually recursively defined with the

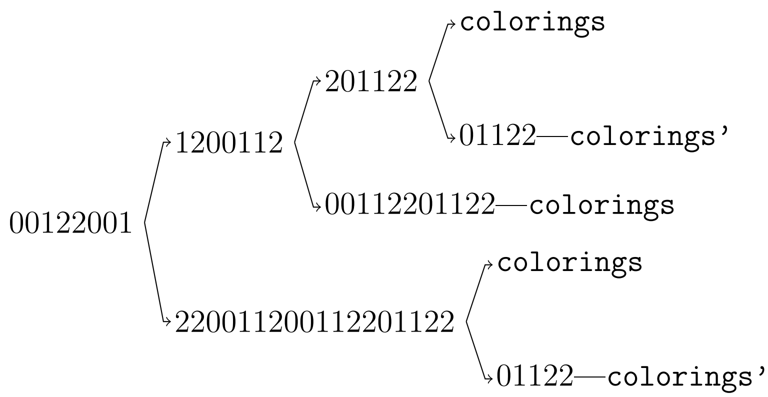

Figure 5.1: The tree `colorings`

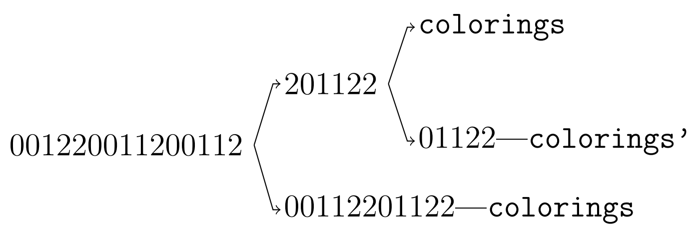

Figure 5.2: The tree `colorings'`

tree `colorings'` shown in Fig. 5.2. Any path of the tree `colorings`, concatenating the color sequence blocks at each vertex, will produce a color sequence of the form $\sum A_i$. Any path producing either a block A_2 or A_4 must be followed by a path producing either A_1, A_2 or A_3. This is is represented by the pruned tree `colorings'`. Conversely, any one way infinite coloring sequence will be contained in a path of `colorings`. Thus it suffices to show that each path in `colorings` contains a partial sum of p for each $p \in X$.

This done by the pair of functions `pathsToLists` and `check`. The function `pathsToLists tree n` creates a list of lists of length n, representing all the paths of length n in `tree`. Then the function `check p` is a Boolean function that, when applied to a list, returns `True` if the list contains a pair of equal elements with indices differing by p. This is equivalent to checking whether the coloring block represented by the list contains a partial sum of p. In this way, running the Haskell code in Appendix C verifies the following theorem.

Theorem 32. *If $p \in X$, then $\{2, 3, 7, 19, p\}$ is class 4.*

CHAPTER 6

Conclusion

6.1 Comparison of methods

In order to show that prime sets of the form $\{2, 3, p\}$ where $p > 5$ are class 3, Eggleton, Erdős and Skilton [12] constructed 3-colorings for those sets. Voigt and Walther [19] also constructed 3-colorings to prove Theorem 10. Thus, in the literature on prime distance graphs, the kappa value has not been used before.

The kappa value has been used previously in order to determine the chromatic number of integer distance graphs where the distance set is not necessarily all primes. The chromatic number of $G(\boldsymbol{Z}, D)$ has been determined when D is a set of 3 integers by Zhu [23] and when D is a set of 4 integers by Liu and Sutedja [16] with the help of the kappa value. In these papers ideas similar to those contained in Lemmas 7 to 9 are used.

In order to establish that prime sets are class 4 the predominate method has been to find subgraphs which are not 3-colorable. This is the method used by Eggleton, Erdős and Skilton [12, 13] to establish that $\{2, 3\} \cup \{p, p + 2\}$ is class 4. The work cited by Voigt which proves that the eight other primes sets of cardinality 4 are class 4 is a chapter in a German book by Walther [20] which I could not locate. The block method used in Chapter 5 is in many ways similar to the method used by Voigt [19] to construct 3-coloring, though I used it negatively to show the impossibility of certain 3-colorings.

6.2 Summary of the main results and future work

The main results of this thesis are Theorems 20 and 32. Together these theorems almost completely classify the prime sets $\{2, 3, 7, p, q\}$. Theorem 32 completely determines the class of the sets $\{2, 3, 7, 19, p\}$, but there are 14 other sets from Theorem 20 that are still undetermined. Proving that each of those is in fact class 4 and showing that condition 3 in Theorem 20 is unnecessary would then complete the classification of $\{2, 3, 7, p, q\}$. We conjecture that the minimal class 4 prime sets $\{2, 3, 7, p, q\}$ are exactly the 31 sets formed by combining $\{2, 3, 7\}$ with one of the pairs listed in Theorem 20. We also propose a stronger conjecture.

Conjecture 33. *A prime set D is class 4 if and only if $\kappa(D) < 1/3$.*

This is a strong conjecture, and the only evidence supporting it is that there are no known counter examples. It seems very hard to prove.

The methods established in Chapter 4 conceivably could be used to classify prime sets of the form $\{2, 3, n, p, q\}$ for primes $n > 7$. This would be interesting in itself, but would not move much closer to a classification of all mimimal class 4 sets with cardinality 5, as there is no bound on n. Indeed Theorem 11 shows that any such bound is unlikely, and the fact that $\kappa(\{2, 3, 179, 191, 199\}) = 22/67 < 1/3$ shows that there would be much work.

The block method developed in Chapter 5 is very tied to the fact that both 7 and 19 are in D. Thus it seems unlikely that those results could be extended to more general prime sets without much further work. In light of this it would be interesting to investigate the distance graphs generated by the known minimal class 4 sets in search of chromatic critical subgraphs, that is subgraphs that are not 3-colorable but the removal of any vertex allows them to be. If it is found that there are only a few such chromatic critical subgraphs over all the known class 4 sets, then determining other prime sets which will generate these subgraphs could be fruitful.

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APPENDIX A

Tables for $\{2,3,7,p,p+6\}$

Table A.1: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+6\}}$ *(Round 2)*

$p \pmod{630}$	$\gcd(p,630)$	$\gcd(p+6,630)$	Point in $I_{\{p,p+6\}}$
5	5		
19		5	
41			43/210
47			1/5
59		5	61/315
61			64/315
73			62/315
79		5	41/210
101			121/630
115	5		41/210
121			61/315
125	5		41/210
131			61/315
145	5		22/105
167			61/315
173			61/315
185	5		62/315
187			1/5
199		5	64/315
205	5		121/630
227			1/5
241			62/315
247			121/630
251			62/315
257			121/630
271			61/315
293			22/105
299		5	121/630
311			
313			
325	5		121/630
331			22/105
353			61/315
367			121/630
373			62/315
377			121/630
383			62/315

Table A.1: *continued*

$p \pmod{630}$	$\gcd(p, 630)$	$\gcd(p+6, 630)$	Point in $I_{\{p,p+6\}}$
397			1/5
419		5	121/630
425	5		64/315
437			1/5
439		5	62/315
451			61/315
457			61/315
479		5	22/105
493			61/315
499		5	41/210
503			61/315
509		5	41/210
523			121/630
545	5		41/210
551			62/315
563			64/315
565	5		61/315
577			1/5
583			43/210
605	5		
619		5	
625	5		
629		5	

APPENDIX B

Tables for $\{2, 3, 7, p, p + 8\}$

Table B.1: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+8\}}$ *(Round 1)*

$p \pmod{126}$	$\gcd(p, 126)$	$\gcd(p + 8, 126)$	Point in $I_{\{p,p+8\}}$
5			
11			
17			3/14
23			
29			
35	7		
41		7	
47			
53			
59			3/14
65			
71			
77	7		
83		7	
89			
95			
101			3/14
107			
113			
119	7		
125		7	

Table B.2: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+8\}}$ *(Round 2)*

$p \pmod{630}$	$\gcd(p, 630)$	$\gcd(p+8, 630)$	Point in $I_{\{p,p+8\}}$
5	5		
11			
23			
29			
47		5	
53			23/105
65	5		
71			139/630
89			139/630
95	5		137/630
107		5	139/630
113			
131			19/90
137		5	
149			23/105
155	5		
173			137/630
179			68/315
191			
197		5	
215	5		
221			19/90
233			
239			
257		5	67/315
263			23/105
275	5		
281			68/315
299			
305	5		
317		5	
323			
341			68/315
347		5	
359			23/105
365	5		67/315
383			
389			
401			19/90
407		5	
425	5		

Table B.2: *continued*

$p \pmod{630}$	$\gcd(p, 630)$	$\gcd(p+8, 630)$	Point in $I_{\{p,p+8\}}$
431			
443			68/315
449			137/630
467		5	
473			23/105
485	5		
491			19/90
509			
515	5		139/630
527		5	137/630
533			139/630
551			139/630
557		5	
569			23/105
575	5		
593			
599			
611			
617		5	

Table B.3: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+8\}}$ *(Round 3)*

$p \pmod{6930}$	$\gcd(p, 6930)$	$\gcd(p+8, 6930)$	Point in $I_{\{p,p+8\}}$
11	11		
23			
29			
113		11	1451/6930
191			295/1386
233			211/990
239			1469/6930
299			295/1386
323			491/2310
383			211/990
389			149/693
431			43/198
509		11	493/2310
593			148/693
599			211/990
611			724/3465
641		11	248/1155
653			247/1155
659			493/2310
743			1489/6930
821			23/110
863			101/462
869	11		12/55
929			163/770
953			1481/6930
1013			724/3465
1019			1487/6930
1061			1487/6930
1139			217/990
1223			146/693
1229			1481/6930
1241			145/693
1271			746/3465
1283			739/3465
1289			247/1155
1373			731/3465
1451			107/495
1493			493/2310
1499		11	1487/6930
1559			1483/6930
1583			293/1386

Table B.3: *continued*

$p \pmod{6930}$	$\gcd(p, 6930)$	$\gcd(p+8, 6930)$	Point in $I_{\{p,p+8\}}$
1643			493/2310
1649			247/1155
1691			739/3465
1769			299/1386
1853			487/2310
1859	11		146/693
1871			106/495
1901			743/3465
1913			106/495
1919			1483/6930
2003			1483/6930
2081			106/495
2123	11		81/385
2129			251/1155
2189	11		1471/6930
2213			211/990
2273			1499/6930
2279			746/3465
2321	11		1459/6930
2399			739/3465
2483			1481/6930
2489		11	81/385
2501			1487/6930
2531			1483/6930
2543			1457/6930
2549			1459/6930
2633			491/2310
2711			499/2310
2753		11	12/55
2759			487/2310
2819		11	493/2310
2843			1469/6930
2903			1453/6930
2909			145/693
2951		11	1481/6930
3029			734/3465
3113	11		106/495
3119			1459/6930
3131			23/110
3161			149/693
3173			148/693

Table B.3: *continued*

$p \pmod{6930}$	$\gcd(p, 6930)$	$\gcd(p+8, 6930)$	Point in $I_{\{p,p+8\}}$
3179	11		731/3465
3263			248/1155
3341			736/3465
3383			733/3465
3389			149/693
3449			
3473			
3533			149/693
3539			733/3465
3581			736/3465
3659			248/1155
3743		11	731/3465
3749			148/693
3761			149/693
3791			23/110
3803			1459/6930
3809		11	106/495
3893			734/3465
3971	11		1481/6930
4013			145/693
4019			1453/6930
4079			1469/6930
4103	11		493/2310
4163			487/2310
4169	11		12/55
4211			499/2310
4289			491/2310
4373			1459/6930
4379			1457/6930
4391			1483/6930
4421			1487/6930
4433	11		81/385
4439			1481/6930
4523			739/3465
4601		11	1459/6930
4643			746/3465
4649			1499/6930
4709			211/990
4733		11	1471/6930
4793			251/1155
4799		11	81/385

Table B.3: *continued*

$p \pmod{6930}$	$\gcd(p, 6930)$	$\gcd(p+8, 6930)$	Point in $I_{\{p,p+8\}}$
4841			106/495
4919			1483/6930
5003			1483/6930
5009			106/495
5021			743/3465
5051			106/495
5063		11	146/693
5069			487/2310
5153			299/1386
5231			739/3465
5273			247/1155
5279			493/2310
5339			293/1386
5363			1483/6930
5423	11		1487/6930
5429			493/2310
5471			107/495
5549			731/3465
5633			247/1155
5639			739/3465
5651			746/3465
5681			145/693
5693			1481/6930
5699			146/693
5783			217/990
5861			1487/6930
5903			1487/6930
5909			724/3465
5969			1481/6930
5993			163/770
6053		11	12/55
6059			101/462
6101			23/110
6179			1489/6930
6263			493/2310
6269			247/1155
6281	11		248/1155
6311			724/3465
6323			211/990
6329			148/693
6413	11		493/2310

Table B.3: *continued*

$p \pmod{6930}$	$\gcd(p, 6930)$	$\gcd(p+8, 6930)$	Point in $I_{\{p,p+8\}}$
6491			43/198
6533			149/693
6539			211/990
6599			491/2310
6623			295/1386
6683			1469/6930
6689			211/990
6731			295/1386
6809	11		1451/6930
6893			
6899			
6911		11	

Table B.4: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+8\}}$ *(Round 4)*

$p \pmod{159390}$	$\gcd(p, 159390)$	$\gcd(p+8, 159390)$	Point in $I_{\{p,p+8\}}$
23	23		
29			
3449			33443/159390
3473	23		11147/53130
6893			3329/15939
6899			16619/79695
6953			33461/159390
6959			6659/31878
10379			33427/159390
10403			33457/159390
13823	23		3347/15939
13829			4781/22770
13883			33463/159390
13889			5548/26565
17309			33433/159390
17333			3041/14490
20753			16642/79695
20759			2374/11385
20813			33479/159390
20819			2377/11385
24239			1013/4830
24263			6691/31878
27683			33289/159390
27689			16733/79695
27743			1108/5313
27749			797/3795
31169			33437/159390
31193			11149/53130
34613			2378/11385
34619			6695/31878
34673			33239/159390
34679			33293/159390
38099			33463/159390
38123			4781/22770
41543			5549/26565
41549			16729/79695
41603			16736/79695
41609			3699/17710
45029			743/3542
45053			33449/159390
48473			11093/53130

Table B.4: *continued*

$p \pmod{159390}$	$\gcd(p, 159390)$	$\gcd(p+8, 159390)$	Point in $I_{\{p,p+8\}}$
48479			1511/7245
48533			3719/17710
48539			33283/159390
51959			4777/22770
51983			6689/31878
55403			317/1518
55409			11159/53130
55463			16738/79695
55469			33277/159390
58889			33461/159390
58913			11141/53130
62333			1849/8855
62339			1583/7590
62393			169/805
62399	23		3328/15939
65819			11153/53130
65843			955/4554
69263			16733/79695
69269			3347/15939
69323			3693/17710
69329			33287/159390
72749	23		531/2530
72773			33431/159390
76193			1513/7245
76199	23		372/1771
76253			33241/159390
76259			6697/31878
79679			
79703			
83123			6697/31878
83129			33241/159390
83183		23	372/1771
83189			1513/7245
86609			33431/159390
86633		23	531/2530
90053			33287/159390
90059			3693/17710
90113			3347/15939
90119			16733/79695
93539			955/4554
93563			11153/53130

Table B.4: *continued*

$p \pmod{159390}$	$\gcd(p, 159390)$	$\gcd(p+8, 159390)$	Point in $I_{\{p,p+8\}}$
96983		23	3328/15939
96989			169/805
97043			1583/7590
97049			1849/8855
100469			11141/53130
100493			33461/159390
103913			33277/159390
103919			16738/79695
103973			11159/53130
103979			317/1518
107399			6689/31878
107423			4777/22770
110843			33283/159390
110849			3719/17710
110903			1511/7245
110909			11093/53130
114329			33449/159390
114353			743/3542
117773			3699/17710
117779			16736/79695
117833			16729/79695
117839			5549/26565
121259			4781/22770
121283			33463/159390
124703			33293/159390
124709			33239/159390
124763			6695/31878
124769			2378/11385
128189			11149/53130
128213			33437/159390
131633			797/3795
131639			1108/5313
131693			16733/79695
131699			33289/159390
135119			6691/31878
135143			1013/4830
138563			2377/11385
138569			33479/159390
138623			2374/11385
138629			16642/79695
142049			3041/14490

Table B.4: *continued*

$p \pmod{159390}$	$\gcd(p, 159390)$	$\gcd(p+8, 159390)$	Point in $I_{\{p,p+8\}}$
142073			33433/159390
145493			5548/26565
145499			33463/159390
145553			4781/22770
145559		23	3347/15939
148979			33457/159390
149003			33427/159390
152423			6659/31878
152429			33461/159390
152483			16619/79695
152489			3329/15939
155909		23	11147/53130
155933			33443/159390
159353			
159359		23	

Table B.5: *Rational points in* $I_{\{2,3,7\}} \cap I_{\{p,p+8\}}$ *(Round 5)*

$p \pmod{4622310}$	$\gcd(p, 4622310)$	$\gcd(p+8, 4622310)$	Point in $I_{\{p,p+8\}}$
29	29		
79679			967963/4622310
79703			322673/1540770
159353			68824/330165
159419			45883/220110
239069			14029/66990
239093			138283/660330
318743			17519/84042
318809			14599/70035
398459			967997/4622310
398483			193597/924462
478133			481774/2311155
478199			107059/513590
557849			967987/4622310
557873	29		3073/14674
637523			192707/924462
637589			481772/2311155
717239			193595/924462
717263			322669/1540770
796913			963533/4622310
796979			160591/770385
876629			967999/4622310
876653			322661/1540770
956303			7647/36685
956369			160588/770385
1036019			322657/1540770
1036043			88001/420210
1115693			41893/200970
1115759			10706/51359
1195409	29		967993/4622310
1195433			29333/140070
1275083			96353/462231
1275149			96352/462231
1354799			968003/4622310
1354823			967979/4622310
1434473			87593/420210
1434539			963527/4622310
1514189			968017/4622310
1514213			64531/308154
1593863			137647/660330
1593929			963521/4622310

Table B.5: *continued*

$p \pmod{4622310}$	$\gcd(p, 4622310)$	$\gcd(p+8, 4622310)$	Point in $I_{\{p,p+8\}}$
1673579			138287/660330
1673603			967973/4622310
1753253	29		481763/2311155
1753319			481762/2311155
1832969			193603/924462
1832993			12571/60030
1912643			481769/2311155
1912709			963541/4622310
1992359			967969/4622310
1992383			107557/513590
2072033			321179/1540770
2072099			481771/2311155
2151749			193601/924462
2151773			107553/513590
2231423			481766/2311155
2231489			963547/4622310
2311139			
2311163			
2390813			963547/4622310
2390879			481766/2311155
2470529			107553/513590
2470553			193601/924462
2550203			481771/2311155
2550269			321179/1540770
2629919			107557/513590
2629943			967969/4622310
2709593			963541/4622310
2709659			481769/2311155
2789309			12571/60030
2789333			193603/924462
2868983			481762/2311155
2869049		29	481763/2311155
2948699			967973/4622310
2948723			138287/660330
3028373			963521/4622310
3028439			137647/660330
3108089			64531/308154
3108113			968017/4622310
3187763			963527/4622310
3187829			87593/420210
3267479			967979/4622310

Table B.5: *continued*

$p \pmod{4622310}$	$\gcd(p, 4622310)$	$\gcd(p+8, 4622310)$	Point in $I_{\{p,p+8\}}$
3267503			968003/4622310
3347153			96352/462231
3347219			96353/462231
3426869			29333/140070
3426893		29	967993/4622310
3506543			10706/51359
3506609			41893/200970
3586259			88001/420210
3586283			322657/1540770
3665933			160588/770385
3665999			7647/36685
3745649			322661/1540770
3745673			967999/4622310
3825323			160591/770385
3825389			963533/4622310
3905039			322669/1540770
3905063			193595/924462
3984713			481772/2311155
3984779			192707/924462
4064429		29	3073/14674
4064453			967987/4622310
4144103			107059/513590
4144169			481774/2311155
4223819			193597/924462
4223843			967997/4622310
4303493			14599/70035
4303559			17519/84042
4383209			138283/660330
4383233			14029/66990
4462883			45883/220110
4462949			68824/330165
4542599			322673/1540770
4542623			967963/4622310
4622273		29	

APPENDIX C

Haskell Code

```
data Tree a = Nil | Node a (Tree a) (Tree a)
  deriving (Show)


data Color = Zero | One | Two
  deriving (Eq, Show)


colorings :: Tree Color
colorings = Node Zero (Node Zero (Node One  (Node Two
           (Node Two  (Node Zero (Node Zero (Node One
            branch_a123  branch_a3'4
                 ) Nil) Nil) Nil) Nil) Nil) Nil) Nil


colorings' = Node Zero (Node Zero (Node One  (Node Two
            (Node Two  (Node Zero (Node Zero (Node One
            branch_a123  Nil
                  ) Nil) Nil) Nil) Nil) Nil) Nil) Nil


branch_a2 = Node Zero (Node One (Node One
            (Node Two (Node Two
            colorings' Nil
```

```
    ) Nil) Nil) Nil) Nil

branch_a1 = Node Two (Node Zero (Node One
            (Node One (Node Two (Node Two
            colorings branch_a2
              ) Nil) Nil) Nil) Nil) Nil

branch_a3 = Node Zero (Node Zero (Node One
           (Node One (Node Two (Node Two
           (Node Zero (Node One  (Node One
           (Node Two (Node Two
            colorings Nil
              ) Nil) Nil) Nil) Nil) Nil)
                Nil) Nil) Nil) Nil) Nil

branch_a123 = Node One (Node Two (Node Zero (Node Zero
             (Node One (Node One (Node Two
              branch_a1 branch_a3
                ) Nil) Nil) Nil) Nil) Nil) Nil

branch_a3'4 = Node Two  (Node Two (Node Zero (Node Zero
             (Node One  (Node One (Node Two  (Node Zero
```

```
             (Node Zero (Node One (Node One  (Node Two
             (Node Two (Node Zero (Node One (Node One
             (Node Two  (Node Two
              colorings branch_a4
               ) Nil) Nil) Nil) Nil) Nil) Nil)
               Nil) Nil) Nil) Nil) Nil) Nil)
               Nil) Nil) Nil) Nil) Nil

branch_a4 = branch_a2

-- a list of all paths of length n from a tree
pathsToLists :: Tree a -> Int -> [[a]]
pathsToLists Nil _ = []
pathsToLists (Node x _ _) 1 = [[x]]
pathsToLists (Node x l r) n =
  (map (x :) (pathsToLists l (n-1))) ++
  (map (x :) (pathsToLists r (n-1)))

-- checks if any two elements distance n apart are equal
check :: Eq a => Int -> [a] -> Bool
check n [] = False
check n (x : xs) = if n < length (x : xs)
```

```
                    && x == (x:xs) !! n
                  then True
                  else check n xs



primes = [31,37,41,43,47,53,67,73,79,83,
          89,109,131,151,157,167,193]


-- a list Bools for each prime,
-- True if every path in colorings contains
-- the prime as a partial sum.
final = map (\p -> all (check p) $
            pathsToLists colorings (p+9)) primes
```