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Eduardo Y. Reynoso
Candidate

Mathematics
Department

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APPROVED: **Dr. Gary Brookfield**
Committee Chairperson
Signature

Dr. Kristin Webster
Faculty Member
Signature

Dr. Anthony Shaheen
Faculty Member
Signature

Dr. Grant Fraser
Department Chairperson
Signature

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ABSTRACT

Reducibility of Symmetric Polynomials

By

Eduardo Y. Reynoso

This thesis investigates the reducibility of trivariate homogeneous symmetric polynomials. For polynomials of degrees 2, 3, 4 and 5 we have complete results. Specifically, we classify all the possible factorizations of such polynomials and give conditions on the coefficients of these polynomials that determine which factorizations occur.

The polynomials of this thesis have indeterminates x, y, and z. The symmetric group $S_{\{x,y,z\}}$ acts on these polynomials by permuting the indeterminates. Symmetric polynomials are left unchanged by this group action. If a polynomial is reducible, then the group acts on the factorization, permuting the factors. The unique factorization property of polynomials allows us to establish which factorizations are possible. Once all possible factorizations are known, we then specify the conditions on the coefficients of the polynomials under which each of these possible factorizations occurs.

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CHAPTER 1

Introduction

In this thesis, we investigate the reducibility of symmetric polynomials in regards to what conditions particular types of polynomials must have in order to be reducible. Throughout this thesis, reducible is interchangeable with factorable. In this thesis all polynomials will be trivariate polynomials, more specifically, polynomials in the polynomial ring $\mathbb{C}[x, y, z]$. Also, all polynomials in this thesis are nonzero unless otherwise stated.

The factorization of multivariate polynomials has been studied for centuries. But most recent research has been focussed on the algorithmic aspects of factorizations with applications to computer algebra systems [2, 4]. Apparently, no other work has been published on the subject of this thesis.

Since we are discussing symmetric polynomials, some notions of group theory and abstract algebra are necessary. This thesis consists of an introductory chapter regarding background information. The factorizations of polynomials of degrees 2, 3, 4 and 5 are in the following chapters respectively.

The concept of a degree of a polynomial in three variables is more complicated than that of a polynomial with one variable. We start by defining the degree of one term of a polynomial in $\mathbb{C}[x, y, z]$ to be the sum of the powers of each distinct variable in that term.

For example, the terms of the polynomial $x^2y + zx + x^3z^2 + 37$ have degrees

$3, 2, 5$, and 0 respectively starting with the first term on the left.

Definition 1.1. *A polynomial is* ***homogeneous*** *if every term in the polynomial has the same degree.*

For example, the polynomial $x^2y^2 + x^2z^2 + y^2z^2$ is homogeneous as each term has degree 4. In contrast, the polynomial $x^2y + zx + x^3z^2 + 37xy + x^2 + z^3$ is not homogeneous as at least two terms in the polynomial have different degrees.

Even so, $x^2y + zx + x^3z^2 + 37 + xy + x^2 + z^3$ can be written as a sum of 4 homogenous polynomials: $(x^3z^2)+(x^2y+z^3)+(zx+xy+x^2)+(37)$. These summands in parentheses are called the homogeneous components of this polynomial.

Definition 1.2. *For each $k = 0, 1, 2, 3, \ldots,$ the* ***degree k homogeneous component*** *of a polynomial f is the sum of all degree k terms of f. Clearly, f is precisely the sum of its nonzero homogeneous components.*

Lemma 1.3. *Let $g, h \in \mathbb{C}[x, y, z]$. If $g \neq 0$, $h \neq 0$ and h is not homogeneous, then gh is not homogeneous.*

Proof. Let g_I and h_I be the homogeneous components of g and h of degree I. Let $g = g_I + g_{I+1} + \cdots + g_J$ be the sum of the homogeneous components of g, where $g_I \neq 0$, $g_J \neq 0$, and $I \leq J$. Let $h = h_K + h_{K+1} + \ldots + h_L$ be the sum of homogeneous components of h with $h_K \neq 0$, $h_L \neq 0$, and $K < L$. Then $gh = g_Ih_K + \cdots + g_Jh_L$, where $I + K < J + L$. (Since h is not homogeneous, $K < L$.) Since $\mathbb{C}[x, y, z]$ is a domain; $g_Ih_K \neq 0$ and $g_Jh_L \neq 0$. So, gh is not homogeneous. □

Theorem 1.4. *Let $f, g, h \in \mathbb{C}[x, y, z]$ be nonzero polynomials such that $f = gh$. Then f is homogeneous if and only if g and h are homogeneous.*

Proof. Clearly, if g is homogeneous of degree M and h is homogeneous of degree N, then f is homogeneous of degree $M + N$. Conversely, if f is homogeneous then by the contrapositive of the previous lemma, g and h are homogeneous. $\square$

By an easy induction argument, if $f = g_1 \cdots g_n$ is a homogeneous polynomial, then g_i is homogeneous for all $i \in \{1, \ldots, n\}$.

Let $S_{\{x,y,z\}}$ be the group of permutations on the set $\{x, y, z\}$. We will use cycle notation for elements of this group, hence

$$S_{\{x,y,z\}} = \{1, (x\ y), (x\ z), (y\ z), (z\ y\ x), (x\ z\ y)\}.$$

Let $\sigma = (x\ y)$ and $\tau = (x\ z)$ be elements of the group $S_{\{x,y,z\}}$. Then $\sigma \circ \tau = (x\ y) \circ (x\ z) = (x\ z\ y)$. Elements of $S_{\{x,y,z\}}$ act on polynomials from $\mathbb{C}[x, y, z]$ by permuting the variables $\{x, y, z\}$. More specifically, for all $\sigma \in S_{\{x,y,z\}}$ and $f \in \mathbb{C}[x, y, z]$, we define

$$\sigma \cdot f(x, y, z) = f(\sigma^{-1}(x), \sigma^{-1}(y), \sigma^{-1}(z)).$$

For example, suppose $\sigma = (z\ y\ x)$, and $g = xy^2 + z^3 + zx^2 \in \mathbb{C}[x, y, z]$. Then $\sigma \cdot g$ is the polynomial created from g by replacing x by y, y by z, and z by x, that is $\sigma \cdot g = yz^2 + x^3 + xy^2$. Similarly, $(x\ y) \cdot g = yx^2 + z^3 + zy^2$.

As an easy consequence of the definition, we have

$$\sigma \cdot (\tau \cdot f) = (\sigma \circ \tau) \cdot f$$

for all $\sigma, \tau \in S_{\{x,y,z\}}$ and $f \in \mathbb{C}[x, y, z]$. Another important property of this group action is that each element of the group acts as a ring homomorphism. Specifically,

$$\sigma \cdot (f + g) = \sigma \cdot f + \sigma \cdot g$$

$$\sigma \cdot (fg) = (\sigma \cdot f)(\sigma \cdot g)$$

$$\sigma \cdot (\lambda g) = \lambda(\sigma \cdot g)$$

for all $f, g \in \mathbb{C}[x, y, z]$, $\sigma \in S_{\{x,y,z\}}$ and $\lambda \in \mathbb{C}$. It is also important to note that, if f is homogeneous then $\sigma \cdot f$ is homogeneous. Also, if f is reducible (irreducible) then so is $\sigma \cdot f$.

A symmetric polynomial in $\mathbb{C}[x, y, z]$ is a polynomial that remains the same after undergoing any permutation of variables in $S_{\{x,y,z\}}$. Specifically, a polynomial $f \in \mathbb{C}[x, y, z]$ is **symmetric** if $\tau \cdot f = f$ for all $\tau \in S_{\{x,y,z\}}$.

Lemma 1.5. *Suppose that $f \in \mathbb{C}[x, y, z]$. Then f is symmetric if and only if $(x\ y) \cdot f = f$ and $(z\ y\ x) \cdot f = f$.*

Proof. If f is symmetric then, by definition, $\tau \cdot f = f$ for all $\tau \in S_{\{x,y,z\}}$, so clearly $(x\ y) \cdot f = f$ and $(z\ y\ x) \cdot f = f$.

Now, we want to show if $(x\ y) \cdot f = f$ and $(z\ y\ x) \cdot f = f$ then f is symmetric. From the given we can infer that, $(y\ z) \cdot f = (z\ y\ x) \circ (x\ y) \cdot f = (z\ y\ x) \cdot f = f$, $(x\ z) \cdot f = (x\ y) \circ (z\ y\ x) \cdot f = (z\ y\ x) \cdot f = f$, as well as $(x\ y\ z) \cdot f = (z\ y\ x) \circ (z\ y\ x) \cdot f = (z\ y\ x) \cdot f = f$. Clearly the identity permutation acting on f is f. Thus every permutation in $S_{\{x,y,z\}}$ sends f to f and f is symmetric. □

Lemma 1.6. *Let $f = gh$, where $f, g, h \in \mathbb{C}[x, y, z]$. If two of these polynomials are symmetric, then all are.*

Proof. Suppose g and h are symmetric. Then as a result, $\tau \cdot g = g$ and $\tau \cdot h = h$ for all $\tau \in S_{\{x,y,z\}}$. Now, $\tau \cdot f = \tau \cdot (gh) = (\tau \cdot g)(\tau \cdot h) = gh = f$ for all $\tau \in S_{\{x,y,z\}}$. Therefore, f is symmetric.

Without loss of generality, suppose g and f are symmetric, so $\tau \cdot g = g$ and

$\tau \cdot f = f$ for all $\tau \in S_{\{x,y,z\}}$. Now $f = \tau \cdot f = \tau \cdot (gh) = (\tau \cdot g)(\tau \cdot h) = g(\tau \cdot h)$ for all $\tau \in S_{\{x,y,z\}}$ and on the other hand $f = gh$. This means $g(\tau \cdot h) = gh$ for all $\tau \in S_{\{x,y,z\}}$. By left cancellation, $\tau \cdot h = h$ for all $\tau \in S_{\{x,y,z\}}$. Hence h is symmetric. $\square$

Definition 1.7. *A polynomial $g \in \mathbb{C}[x, y, z]$ is* ***almost symmetric*** *if, for all $\sigma \in S_{\{x,y,z\}}$, $\sigma \cdot g = \lambda_\sigma g$ for some $\lambda_\sigma \in \mathbb{C}^\times$.*

Lemma 1.8. *If g is almost symmetric, as in Definition 1.7, then $\phi : S_{\{x,y,z\}} \to \mathbb{C}^\times$, defined by $\phi(\sigma) = \lambda_\sigma$, for $\sigma \in S_{\{x,y,z\}}$, is a group homomorphism.*

Proof. We want to show that $\phi(\sigma \circ \tau) = \phi(\sigma)\phi(\tau)$ for all $\sigma, \tau \in S_{\{x,y,z\}}$ or equivalently, since $\sigma \cdot g = \lambda_\sigma g$ for all $\sigma \in S_{\{x,y,z\}}$, $\lambda_{\sigma\circ\tau} = \lambda_\sigma \lambda_\tau$. We know that $(\sigma \circ \tau)g = \sigma \cdot (\tau \cdot g)$. Well, $(\sigma \circ \tau) \cdot g = \lambda_{\sigma\circ\tau} g$ and $(\sigma \circ \tau)g = \sigma \cdot (\tau \cdot g) = \sigma \cdot (\lambda_\tau g) = \lambda_\tau \sigma \cdot g = \lambda_\tau \lambda_\sigma g$ for all $\sigma, \tau \in S_{\{x,y,z\}}$. The previous sentence implies that $\lambda_{\sigma\circ\tau} g = \lambda_\tau \lambda_\sigma g$ for all $\sigma, \tau \in S_{\{x,y,z\}}$ and by right cancelation of g, $\lambda_{\sigma\circ\tau} = \lambda_\tau \lambda_\sigma$ or $\lambda_{\sigma\circ\tau} = \lambda_\sigma \lambda_\tau$ as $\lambda_\tau, \lambda_\sigma \in \mathbb{C}^\times$ for all $\sigma, \tau \in S_{\{x,y,z\}}$. Therefore ϕ is a group homomorphism. $\square$

Theorem 1.9. *If $g \in \mathbb{C}[x, y, z]$ is almost symmetric then:*

(1) *g is symmetric OR*

(2) *$g = (x - y)(x - z)(y - z)h$ where $h \in \mathbb{C}[x, y, z]$ is symmetric*

Proof. By Lemma 1.8, $\phi : S_{\{x,y,z\}} \to \mathbb{C}^\times$, defined by $\phi(\sigma) = \lambda_\sigma$ is a group homomorphism.

Case I: If $\ker \phi = S_{\{x,y,z\}}$, then $\phi(\sigma) = 1$ for all $\sigma \in S_{\{x,y,z\}}$, then g is symmetric.

Case II: If $\ker \phi = \{1\}$, then $S_{\{x,y,z\}} \cong \phi(S_{\{x,y,z\}}) \leq \mathbb{C}^\times$, but $S_{\{x,y,z\}}$ is not abelian, so this case cannot happen.

Case III: Suppose $\ker \phi = \{1, (z\ y\ x), (x\ y\ z)\}$. This implies that $\phi(S_{\{x,y,z\}}) \cong \mathbb{Z}_2$.

As a result, $\phi(S_{\{x,y,z\}}) = \{1, -1\}$. Then, $\lambda_\sigma = 1$ if σ is even and -1 otherwise. A polynomial with this property is called antisymmetric.

Temporarily view g as a polynomial in z with coefficients from $\mathbb{C}[x, y]$, or in other words, $g \in \mathbb{C}[x, y][z]$. The equation $(z\ x) \cdot g = -g$ implies $g(z, y, x) = -g(x, y, z)$. Setting $z = x$ gives $g(x, y, x) = -g(x, y, x)$ which implies that $g(x, y, x) = 0$. This means that x is a root of $g(x, y, z) \in \mathbb{C}[x, y][z]$.

Since x is a root of $g(x, y, z) \in \mathbb{C}[x, y][z]$, $z-x$ is a factor of $g(x, y, z)$. Similarly, $x - y$, and $y - z$ are factors of $g \in \mathbb{C}[x, y, z]$. As $x - y$, $z - x$ and, $y - z$ are relatively prime, $g = (x - y)(x - z)(y - z)h$ for some $h \in \mathbb{C}[x, y, z]$.

Now, notice that $-g = -(x - y)(x - z)(y - z)h$ and at the same time; $-g = (x\ y) \cdot g = -(x-y)(x-z)(y-z)(x\ y) \cdot h$. So, $-(x-y)(x-z)(y-z)h = -(x-y)(x-z)(y - z)(x\ y) \cdot h$ and by left cancellation, $h = (x\ y) \cdot h$. Similarly, $g = (z\ y\ x) \cdot g = (x - y)(x - z)(y - z)(z\ y\ x) \cdot h$ and $g = (x - y)(x - z)(y - z)h$ imply $h = (z\ y\ x) \cdot h$. Hence, h is symmetric by Lemma 1.5. $\square$

Corollary 1.10. *If g has degree less than 3 and is almost symmetric, then it is symmetric.*

Proof. An immediate consequence of Theorem 1.9. $\square$

Since $\mathbb{C}$ is a unique factorization domain, so is the polynomial ring $\mathbb{C}[x, y, z]$. For the proof, see [3, Theorem 6.14]. In this context, unique factorization means that any nonzero polynomial in $\mathbb{C}[x, y, z]$ can be written as a product of irreducible polynomials, unique up to multiplication by nonzero complex numbers.

We use the notation $\mathbb{C}^\times f$ to represent the set of all nonzero scalar multiples of the polynomial f.

Lemma 1.11. *Suppose $f = g_1 g_2 \cdots g_n$ is almost symmetric and each $g_i \in \mathbb{C}[x, y, z]$ is irreducible for $i = 1, 2, \ldots, n$. Let $T = \{\mathbb{C}^\times g_1, \mathbb{C}^\times g_2, \ldots, \mathbb{C}^\times g_n\}$, then there is a group homomorphism $\phi : S_{\{x,y,z\}} \to S_T$ defined by $\sigma \cdot (\mathbb{C}^\times g_i) = \mathbb{C}^\times(\sigma \cdot g_i)$ for $i = 1, 2, ..., n$.*

Proof. First we must check that $\mathbb{C}^\times(\sigma \cdot g_i)$ is in T. Since f is almost symmetric, $\sigma \cdot f = \lambda f$ for some $\lambda \in \mathbb{C}$ and,

$$\lambda f = \sigma \cdot f = \sigma \cdot (g_1 g_2 \ldots g_k) = (\sigma \cdot g_1)(\sigma \cdot g_2) \cdots (\sigma \cdot g_n)$$

is a factorization of f into irreducibles. By unique factorization, $\sigma \cdot g_i$ is a nonzero scalar multiple of g_j for some j and so $\mathbb{C}^\times(\sigma \cdot g_i) = \mathbb{C}^\times g_j \in T$. Thus, $\mathbb{C}^\times(\sigma \cdot g_i) \in T$ and ϕ is well-defined.

Next, we confirm that ϕ is a group homomorphism. Now, $(\sigma \circ \tau) \cdot (\mathbb{C}^\times g_i) = \mathbb{C}^\times((\sigma \circ \tau) \cdot g_i) = \mathbb{C}^\times \sigma \cdot (\tau \cdot g_i) = (\sigma) \cdot (\mathbb{C}^\times(\tau \cdot g_i)) = ((\sigma)((\tau)) \circ \mathbb{C}^\times g_i)$ for all $i \leq n$. So, $\phi(\sigma \circ \tau) = \phi(\sigma) \circ \phi(\tau)$ for all $\sigma, \tau \in S_{\{x,y,z\}}$ and ϕ is a group homomorphism. $\square$

Theorem 1.12. *Suppose $f = g_1 g_2 \cdots g_n$ is almost symmetric and each g_i is irreducible for $i = 1, 2, \ldots, n$. Write $f = G_1 G_2 \cdots G_N$, where G_k is the product of all degree k irreducible factors of f in the above factorization, (or $G_k = 1$ if no such factors exist). Then G_k is almost symmetric for all $k \in \{1, \ldots, N\}$.*

Proof. Let $\sigma \in S_{\{x,y,z\}}$ be arbitrary. Since f is almost symmetric, then $\sigma \cdot f = \lambda f$ for some $\lambda \in \mathbb{C}$. Then $\lambda g_1 \cdots g_n = \lambda f = \sigma \cdot f = \sigma \cdot (g_1 \cdots g_n) = (\sigma \cdot g_1) \cdots (\sigma \cdot g_n)$. For each i, $\sigma \cdot g_i$ is an irreducible polynomial, so by unique factorization, $\sigma \cdot g_i$ is a scalar multiple of g_j for some $j \in \{1, \cdots, n\}$. Moreover, if g_i has degree k, then so does g_j.

Hence, σ acts by permuting the factors of G_k, and so, $\sigma \cdot G_k$ is a scalar multiple of G_k. Since this holds for all $\sigma \in S_{\{x,y,z\}}$, G_k is almost symmetric. $\square$

Corollary 1.13. *Suppose f factors as described in Theorem 1.12. If f is symmetric, then G_k is symmetric for all $k \in \{1, \ldots, N\}$.*

Proof. By Theorem 1.12, G_k is almost symmetric for each $k \in \{1, \ldots, N\}$. Then by Theorem 1.9, either G_k is symmetric or has $(x-y)(x-z)(y-z)$ as a factor. For $k \geq 2$, G_k has no degree one factors, and so G_k is symmetric. Since $f = G_1 G_2 \cdots G_N$, and f and $G_2 \cdots G_N$ are symmetric, then, by Lemma 1.6, G_1 is also symmetric. □

Let's see how this applies to the simplest case, the degree one polynomials.

Lemma 1.14. *Suppose $g \in \mathbb{C}[x, y, z]$ has $\deg g = 1$.*

1. *If $(z\ y\ x) \cdot g = \lambda g$ for some $\lambda \in \mathbb{C}$, then $\lambda^3 = 1$ and $g = a(x + \lambda^2 y + \lambda z)$ for some $a \in \mathbb{C}$.*

2. *If g is almost symmetric, then g is symmetric and $g = a(x+y+z)$ for some $a \in \mathbb{C}$.*

Proof. (1). Suppose $(z\ y\ x) \cdot g = \lambda g$. Since $(z\ y\ x)^3 = 1 \in S_{\{x,y,z\}}$, we have $g = 1 \cdot g = (z\ y\ x)^3 \cdot g = \lambda^3 g$. By cancelation, $\lambda^3 = 1$. Let $g = ax + by + cz$ for some $a, b, c \in \mathbb{C}$. Suppose $(z\ y\ x) \cdot g = \lambda g$. Then, $ay + bz + cx = \lambda(ax + by + cz)$ and matching coefficients, $c = \lambda a$, $a = \lambda b$ and $b = \lambda c$. Since $b = \lambda c = \lambda(\lambda a) = \lambda^2 a$, we get that $g = a(x + \lambda^2 y + \lambda z)$.

(2). Since g is almost symmetric and has degree 1, by Corollary 1.10, g is symmetric. In particular, $(z\ y\ x) \cdot g = g$. By (1), with $\lambda = 1$, we get $g = a(x + y + z)$ for some $a \in \mathbb{C}$. □

CHAPTER 2

Factorization of Degree Two Symmetric Polynomials

In this chapter we will explore the factorization of symmetric, trivariate, homogeneous, degree 2 polynomials. In particular, we get the first instance of a symmetric polynomial that factors into a product of two polynomials that are not symmetric.

Lemma 2.1. *Let f be a homogeneous polynomial of degree 2. Suppose $(z\ y\ x)\cdot f = \lambda f$ for some $\lambda \in \mathbb{C}$. Then $\lambda^3 = 1$ and $f = a(x^2 + \lambda^2 y^2 + \lambda z^2) + b(xy + \lambda^2 yz + \lambda xz)$ for some $a, b \in \mathbb{C}$.*

Proof. Let $f = a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5yz + a_6xz$ where $a_1, a_2, a_3, a_4, a_5, a_6 \in \mathbb{C}$ and suppose $(z\ x\ y)\cdot f = \lambda f$. Now, $(z\ y\ x)\cdot f = a_1y^2 + a_2z^2 + a_3x^2 + a_4yz + a_5zx + a_6yx$ and $\lambda f = \lambda(a_1x^2 + a_2y^2 + a_3z^2 + a_4xy + a_5yz + a_6xz)$. As $(z\ y\ x)\cdot f = \lambda f$, equating coefficients yields $a_1 = \lambda a_2$, $a_2 = \lambda a_3$, $a_3 = \lambda a_1$, $a_4 = \lambda a_5$, $a_5 = \lambda a_6$, $a_6 = \lambda a_4$. These equations imply that, $a_3 = \lambda a_1$, $a_2 = \lambda^2 a_1$, $a_6 = \lambda a_4$, and $a_5 = \lambda^2 a_4$. Therefore, $f = a_1(x^2 + \lambda^2y^2 + \lambda z^2) + a_4(xy + \lambda^2 yz + \lambda xz)$ and by letting $a_1 = a$ and $a_4 = b$ we get that, $f = a(x^2 + \lambda^2 y^2 + \lambda z^2) + b(xy + \lambda^2 yz + \lambda xz)$.

Consider $(z\ y\ x)\cdot f = \lambda f$. Now, $(z\ y\ x)^3 \cdot f = \lambda^3 f$ implies that $f = \lambda^3 f$ and hence $\lambda^3 = 1$. □

Lemma 2.2. *If f is a symmetric homogeneous degree 2 polynomial, then*

$$f = a(x^2 + y^2 + z^2) + b(xy + yz + xz) \tag{2.1}$$

for some $a, b \in \mathbb{C}$.

Proof. Since f is symmetric then $(z\ y\ x)\cdot f = f$ and by Lemma 2.1 with $\lambda = 1$,

$f = a(x^2 + y^2 + z^2) + b(xy + yz + xz)$ for some $a, b \in \mathbb{C}$. □

Theorem 2.3. *Suppose f is a symmetric homogeneous degree 2 polynomial with the form $f = a(x^2 + y^2 + z^2) + b(xy + yz + xz)$ as in Lemma 2.2.*

(1) *If $b = 2a$, then $f = a(x + y + z)^2$*

(2) *If $b = -a$ then, $f = a(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$, where $\omega = e^{2\pi i/3}$*

Proof. (1) Suppose $b = 2a$. Then, $f = a(x^2 + y^2 + z^2) + 2a(xy + yz + xz)$ and this polynomial factors as $a(x + y + z)^2$.

(2) Suppose $b = -a$. Then $f = -a(xy + xz + yz) + a(x^2 + y^2 + z^2)$ and an easy calculation shows that $f = a(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$ where $\omega = e^{2\pi i/3}$.

□

Example 2.4. *The special case $a = 1$, $b = -1$ gives the factorization*

$$f = (x^2 + y^2 + z^2) - (xy + yz + xz) = (x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$$

where $\omega = e^{2\pi i/3}$. Notice that f is a symmetric polynomial whose factors are not symmetric.

Theorem 2.5. *Suppose f is a symmetric homogeneous degree 2 polynomial with the form $f = a(x^2 + y^2 + z^2) + b(xy + yz + xz)$ as in Lemma 2.2. Then, f is reducible if and only if $b = 2a$ or $b = -a$, if and only if*

$$f = a(x + \lambda y + \lambda^2 z)(x + \lambda^2 y + \lambda z), \tag{2.2}$$

where $\lambda \in \mathbb{C}^\times$ and $\lambda^3 = 1$.

Proof. From Theorem 2.3, it is easy to see that the condition, $b = 2a$ or $b = -a$, is equivalent to f having the form of equation (2.2). Just as obvious, if either of these conditions is true, then f is reducible.

Suppose f is reducible. Then $f = hk$ where $\deg h = \deg k = 1$ and h and k are homogeneous by Theorem 1.4. Let $T = \{\mathbb{C}^\times h, \mathbb{C}^\times k\}$, and let $\phi : S_{\{x,y,z\}} \to S_T$ be the group homomorphism as in Lemma 1.11. We then have two cases to consider:

(1) Suppose $\ker\phi = S_{\{x,y,z\}}$. Then, for all $\sigma \in S_{\{x,y,z\}}$, $\sigma \cdot h = \lambda_\sigma h$ and $\sigma \cdot k = \mu_\sigma k$ for some $\lambda_\sigma, \mu_\sigma \in \mathbb{C}^\times$, that is, h and k are almost symmetric. By Lemma 1.14, $h = \alpha(x + y + z)$ and $k = \beta(x + y + z)$ for some $\alpha, \beta \in \mathbb{C}^\times$. Therefore, $f = a(x+y+z)^2$ where $a = \alpha\beta$ by matching coefficients. So, f is a special case of (2.2) where $\lambda = 1$.

(2) Suppose $\ker\phi \neq S_{\{x,y,z\}}$. This means that the image of ϕ is not the trivial subgroup of S_T which implies that the cardinality of T is 2. Or in other words, that h and k are linearly independent. The only subgroup of $S_{\{x,y,z\}}$ that has index 2 is $\{1, (z\ y\ x), (x\ y\ z)\}$, so this subgroup is the kernel of ϕ. Because $(z\ y\ x) \in \ker\phi$, then $(z\ y\ x)\cdot h = \lambda_1 h$ and $(z\ y\ x)\cdot k = \lambda_2 k$ for some $\lambda_1, \lambda_2 \in \mathbb{C}$. Since $\deg h = \deg k = 1$, by Lemma 1.14, $h = \alpha(x + \lambda_1^2 y + \lambda_1 z)$ and $k = \beta(x + \lambda_2^2 y + \lambda_2 z)$ for some $\alpha, \beta, \lambda_1, \lambda_2 \in \mathbb{C}^\times$ with $\lambda_1^3 = \lambda_2^3 = 1$.

Because $hk = f = (z\ y\ x)\cdot f = (z\ y\ x)\cdot(hk) = ((z\ y\ x)\cdot h)((z\ y\ x)\cdot k) = \lambda_1 h \lambda_2 k = \lambda_1\lambda_2 hk$, we get $\lambda_1\lambda_2 = 1$. Canceling λ_1 from $\lambda_1\lambda_2 = \lambda_1^3$ we get $\lambda_2 = \lambda_1^2$; and so $k = \beta(x + \lambda_1 y + \lambda_1^2 z)$.

If we set $\lambda_1 = \lambda$, we can write $f = a(x + \lambda y + \lambda^2 z)(x + \lambda^2 y + \lambda z)$, with $\lambda^3 = 1$ and, by matching coefficients, $\alpha\beta = a$. $\square$

CHAPTER 3

Factorization of Degree Three Symmetric Polynomials

In this chapter we will determine the factorization of degree three, trivariate, homogeneous, symmetric polynomials.

Lemma 3.1. *A polynomial f is homogeneous, symmetric and has degree* 3 *if and only if it has the form:*

$$f = a(x^3 + y^3 + z^3) + b(xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2) + cxyz,$$

where $a, b, c \in \mathbb{C}$.

Proof. If f is a homogeneous degree 3 polynomial, it can be written as

$$f = c_1x^3 + c_2y^3 + c_3z^3 + c_4xy^2 + c_5xz^2 + c_6yx^2 + c_7yz^2 + c_8zx^2$$

$$+c_9zy^2 + c_{10}xyz$$

for some $c_1, c_2, \ldots, c_{10} \in \mathbb{C}$. If, in addition, f is symmetric, then $(x\ y)\cdot f = (z\ y\ x)\cdot f = f$, that is,

$$f = (x\ y) \cdot f = c_1y^3 + c_2x^3 + c_3z^3 + c_4yx^2 + c_5yz^2 + c_6xy^2 + c_7xz^2 + c_8zy^2$$

$$+c_9zx^2 + c_{10}xyz.$$

$$f = (x\ y\ z) \cdot f = c_1y^3 + c_2z^3 + c_3x^3 + c_4yz^2 + c_5yx^2 + c_6zy^2 + c_7zx^2 + c_8xy^2$$

$$+c_9xz^2 + c_{10}xyz.$$

Matching coefficients we obtain $c_1 = c_2 = c_3$ and $c_4 = c_5 = c_6 = c_7 = c_8 = c_9$.

Consequently, $f = c_1(x^3+y^3+z^3)+c_4(xy^2+xz^2+yx^2+yz^2+zx^2+zy^2)+c_{10}xyz$, and by letting $c_1 = a$, $c_4 = b$ and $c_{10} = c$, f has the claimed form. Conversely, if $f = a(x^3 + y^3 + z^3) + b(xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2) + cxyz$, where $a, b, c \in \mathbb{C}$, then f is a degree 3 symmetric homogeneous polynomial. $\square$

Theorem 3.2. *Let f be a trivariate, homogeneous, symmetric, degree 3 polynomial. Suppose $f = ghk$ with $\deg g = \deg h = \deg k = 1$. Then at least one of the following occurs:*

(1) $f = a(x + y + z)(x + \lambda y + \lambda^2 z)(x + \lambda^2 y + \lambda z)$ *for some* $a, \lambda \in \mathbb{C}$ *with* $\lambda^3 = 1$.

(2) $f = C(A(x + y) + Bz)(A(x + z) + By)(A(y + z) + Bx)$ *for some* $A, B, C \in \mathbb{C}$.

Proof. By the unique factorization theorem of polynomials we get a group homomorphism $\phi : S_{\{x,y,z\}} \to S_T$ where $T = \{\mathbb{C}^\times g, \mathbb{C}^\times h, \mathbb{C}^\times k\}$ as in Lemma 1.11. The proof now splits into three cases depending on the kernel of ϕ.

(1) Suppose that $\ker \phi = S_{\{x,y,z\}}$. This means that every element in $S_{\{x,y,z\}}$ gets sent to the identity element of S_T. This means that $\sigma \cdot g = \lambda_\sigma g$ for all $\sigma \in S_{\{x,y,z\}}$ and, by Lemma 1.14, g is a scalar multiple of $x + y + z$. Similarly, h and k are also scalar multiples of $x + y + z$. This implies that $f = a(x + y + z)^3$, for some $a \in \mathbb{C}$, a special case of factorization (1) with $\lambda = 1$, and also a special case of factorization (2) with $A = B = 1$ and $C = a$.

(2) Suppose that $\ker \phi = \{1, (z\ y\ x), (x\ y\ z)\}$. This implies that $|\operatorname{im} \phi| = 2$, and $\operatorname{im} \phi = \{1_T, \mu\}$ where $\mu \in S_T$ and $|\mu| = 2$. Without loss of generality, $\operatorname{im} \phi = \{1, (\mathbb{C}^\times g, \mathbb{C}^\times h)\}$. Since $\mathbb{C}^\times k$ is fixed by $S_{\{x,y,z\}}$, this implies that k is a scalar

multiple of $x+y+z$ by Lemma 1.14. Since f is symmetric and k is symmetric, then, by Lemma 1.6, gh is a symmetric polynomial. By construction, gh is reducible. Since gh is a symmetric homogeneous degree 2 polynomial, then by Theorem 2.5, gh is a scalar multiple of $(x+\lambda y+\lambda^2 z)(x+\lambda^2 y+\lambda z)$ for some $\lambda \in \mathbb{C}$ with $\lambda^3 = 1$. Therefore, $f = a(x+y+z)(x+\lambda y+\lambda^2 z)(x+\lambda^2 y+\lambda z)$, for some $a \in \mathbb{C}$ and $\lambda^3 = 1$, as in factorization (1).

(3) Suppose that $\ker\phi = \{1\}$. As $\ker\phi = \{1\}$, we have $\phi(S_{\{x,y,z\}}) \cong S_3$. Let $\sigma = (z\ y\ x)$. Since σ has order 3, it acts as an element of order 3 in S_T. Without loss of generality, $\sigma \cdot g = \lambda_1 h$, $\sigma \cdot h = \lambda_2 k$ and $\sigma \cdot k = \lambda_3 g$ for some $\lambda_1, \lambda_2, \lambda_3 \in \mathbb{C}$.

Let $g = \alpha x + \beta y + \gamma z$ for some $\alpha, \beta, \gamma \in \mathbb{C}$. Now, $\sigma \cdot g = \alpha y + \beta z + \gamma x$, and so $h = \frac{1}{\lambda_1}(\alpha y+\beta z+\gamma x)$. Similarly, $\sigma \cdot h = \frac{1}{\lambda_1}(\alpha z+\beta x+\gamma y)$, so $k = \frac{1}{\lambda_1\lambda_2}(\alpha z+\beta x+\gamma y)$ and $\sigma \cdot k = \frac{1}{\lambda_1\lambda_2}(\alpha x+\beta y+\gamma z)$. Since f is symmetric, the coefficients of $y^2 z$ and $z^2 y$ are equal. By expanding $f = ghk$ and comparing those coefficients, we get the equation $(\alpha-\beta)(\alpha-\gamma)(\beta-\gamma) = 0$. By the zero product rule, $\alpha = \beta, \alpha = \gamma$ or $\beta = \gamma$.

If $\alpha = \beta$, then

$$g = \alpha(x+y) + \gamma z$$
$$h = \frac{1}{\lambda_1}(\alpha(y+z) + \gamma x)$$
$$k = \frac{1}{\lambda_1\lambda_2}(\alpha(z+x) + \gamma y)$$

So $f = ghk$ has the form of factorization (2). The other two possibilities, $\alpha = \gamma$ and $\beta = \gamma$, lead to a factorization of f of the same form. □

For the proof of the main theorem of this chapter, it is useful to write symmetric degree three polynomials in a different form than f in Lemma 3.1. This new form is introduced in the following lemma.

Notice that the form of a homogeneous symmetric degree 3 polynomial in the upcoming Lemma 3.3 is different than the form in Lemma 3.1. The reason we do this is because with Theorem 3.4 is easier to prove with the form of Lemma 3.3.

Lemma 3.3. *A polynomial f is homogeneous, symmetric and has degree 3 if and only if it has the form:*

$$f = a_0(x+y+z)^3 + b_0(x+y)(x+z)(y+z) + c_0xyz, \tag{3.1}$$

for some $a_0, b_0, c_0 \in \mathbb{C}$.

Proof. By Lemma 3.1, a polynomial f is homogeneous, symmetric and has degree 3 if and only if it has the form: $f = a(x^3+y^3+z^3) + b(xy^2+xz^2+yx^2+yz^2+zx^2+zy^2) + cxyz$, for some $a, b, c \in \mathbb{C}$. So, it suffices to show that a polynomial f has the form of Lemma 3.1 if it has the form of (3.1).

Expanding (3.1) yields, $f = a_0(x^3+y^3+z^3) + (3a_0+b_0)(y^2z+yz^2+x^2y+x^2z+xy^2+xz^2) + (6a_0+2b_0+c_0)xyz$. By matching coefficients with f written as in Lemma 3.1 we get:

$$a = a_0$$

$$b = 3a_0 + b_0$$

$$c = 6a_0 + 2b_0 + c_0$$

These equations can be solved for a_0, b_0 and c_0.

$$\begin{aligned} a_0 &= a \\ b_0 &= b - 3a \\ c_0 &= c - 2b \end{aligned} \tag{3.2}$$

Because of these relationships between the coefficients, any polynomial of the form in Lemma 3.1 can be written in the form of (3.1) and vice versa. $\square$

For the next theorem, set $R_0 = (b_0 - c_0)^2 a_0 - c_0 b_0^2$.

Theorem 3.4. *Let $f = a_0(x+y+z)^3 + b_0(x+y)(x+z)(y+z) + c_0xyz$. Then f is reducible if and only if one of the following is true:*

(1) $b_0 = c_0$. *In this case,*

$$f = (x+y+z)\left(a_0(x^2+y^2+z^2) + (2a_0+b_0)(xy+xz+yz)\right). \tag{3.3}$$

(2) $R_0 = 0$ *and* $b_0 \neq c_0$. *In this case,*

$$f = \frac{1}{(b_0-c_0)^2}(c_0x + b_0(y+z))(c_0y + b_0(x+z))(c_0z + b_0(x+y)) \tag{3.4}$$

Proof. If $b_0 = c_0$, then an easy calculation gives (3.3).

Now suppose $R_0 = 0$ and $b_0 \neq c_0$. Solving the equation $R_0 = 0$ for a_0 gives $a_0 = c_0 b_0^2/(b_0 - c_0)^2$. Substituting this value into f and factoring gives (3.4). In each of these cases f is reducible.

Now, let's suppose that f is reducible. In this case f must either factor into three degree one polynomials or into a degree one polynomial and an irreducible degree two polynomial ($f = ghk$, where $\deg g = \deg h = \deg k = 1$ or $f = gh$ where

$\deg g = 1$ and $\deg h = 2$ respectively). So let's do this in cases. To prove these cases it is important to note that f is symmetric because of Lemma 3.3.

Case I: Let $f = gh$ where h is irreducible, $\deg g = 1$ and $\deg h = 2$. By Corollary 1.13, g and h are symmetric.

By Lemma 1.14, $g = \alpha(x + y + z)$ for some $\alpha \in \mathbb{C}$. By Lemma 2.2, $h = B(xy + xz + yz) + C(x^2 + y^2 + z^2)$ for some $B, C \in \mathbb{C}$. Now, by matching coefficients in $f = gh$, we get $a_0 = \alpha C$ and $b_0 = c_0 = \alpha(B - 2C)$, so f factors as in (3.3).

Case II: Let $f = ghk$ where $\deg g = \deg h = \deg k = 1$. By Theorem 3.2, then

$$f = a(x + y + z)(x + \lambda y + \lambda^2 z)(x + \lambda^2 y + \lambda z)$$

or

$$f = C(A(x + y) + Bz)(A(x + z) + By)(A(y + z) + Bx).$$

If the first of these factorizations occurs we find, by equating coefficients with f as in (3.1), $b_0 = c_0 = a(\lambda + \lambda^2 - 2)$, so f factors as in (1).

If the second factorization occurs, by equating coefficients with f as in (3.1), we get $a_0 = A^2BC$, $b_0 = A^3C - 2A^2BC + AB^2C$, and $c_0 = A^2BC - 2AB^2C + B^3C$. Plugging these conditions into R_0 we get zero. Since $b_0 - c_0 = (A - B)^3C$, f factors as in (2), except when $A = B$. However, if $A = B$, then $f = CA^3(x + y + z)^3$, which is a special case of (1). $\square$

For the next theorem, let $R = 9a^3 - 3ab^2 + 2b^3 - 3a^2c - b^2c + ac^2$.

Theorem 3.5. *Let* $f = a(x^3 + y^3 + z^3) + b(x^2y + xy^2 + x^2z + y^2z + xz^2 + yz^2) + cxyz$ *for some* $a, b, c \in \mathbb{C}$. *Then* f *is reducible if and only if one of the following occurs:*

(1) $3b - 3a - c = 0$. *In this case,*

$$f = (x + y + z)\left(a(x^2 + y^2 + z^2) + (b - a)(xy + xz + yz)\right).$$

(2) $R = 0$ *and* $3b - 3a - c \neq 0$. *In this case,*

$$\begin{aligned} f = &\frac{1}{(3b - 3a - c)^2}\big((c - 2b)x + (b - 3a)(y + z)\big) \\ &\times \big((c - 2b)y + (b - 3a)(x + z)\big)\big((c - 2b)z + (b - 3a)(x + y)\big). \end{aligned}$$

Proof. This is Theorem 3.4 applied to polynomials written in the form of Lemma 3.1, by the coefficient relationship in (3.2). □

Example 3.6. *Consider the polynomial,* $f = x^2y + xy^2 + x^2z + y^2z + xz^2 + yz^2 + 3xyz$, *where* $a = 0$, $b = 1$ *and* $c = 3$. *Since* $3b - 3a - c = 3 - 0 - 3 = 0$, *by Theorem 3.4,* f *factors as* $(x + y + z)(xy + xz + yz)$.

Corollary 3.7. *In the context of Theorem 3.5, suppose that* $3b - 3a - c = 0$ *and* $f = (x + y + z)(a(x^2 + y^2 + z^2) + (b - a)(xy + xz + yz))$. *Then* f *is a product of three linear factors if and only if one of the following occurs:*

(1) $b = 3a$ *and* $c = 6a$. *In this case,* $f = a(x + y + z)^3$.

(2) $b = 0$ *and* $c = -3a$. *In this case,* $f = a(x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$, *where* $\omega = e^{2\pi i/3}$.

Proof. Since $f = (x + y + z)(a(x^2 + y^2 + z^2) + (a - b)(xy + xz + yz))$ to see if this polynomial factors further it suffices to check if

$$g = \left(a(x^2 + y^2 + z^2) + (b - a)(xy + xz + yz)\right)$$

is reducible. Theorem 2.5 tells us that $(a(x^2+y^2+z^2)+(b-a)(xy+xz+yz))$ is reducible if and only if one of the following occurs: $b=3a$ then $g=a(x+y+z)^2$ or $b=0$ then $g=a(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$, where $\omega=e^{2\pi i/3}$. This makes the claim clear, that if $b=3a$ and $c=6a$ then $f=a(x+y+z)^3$ or if $b=0$ and $c=-3a$ then $f=a(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$, where $\omega=e^{2\pi i/3}$. $\square$

Combining the previous corollary and Theorem 3.5 we see that $f=a(x^3+y^3+z^3)+b(x^2y+xy^2+x^2z+y^2z+xz^2+yz^2)+cxyz$ is completely reducible if and only if certain conditions apply. These conditions will be described in the next theorem. For the next theorem, let $R=9a^3-3ab^2+2b^3-3a^2c-b^2c+ac^2$

Theorem 3.8. *Let $f=a(x^3+y^3+z^3)+b(x^2y+xy^2+x^2z+y^2z+xz^2+yz^2)+cxyz$ for some $a,b,c\in\mathbb{C}$. Then f is completely reducible if and only if one of the following occurs:*

(1) *$b=3a$ and $c=6a$. In this case, $f=a(x+y+z)^3$.*

(2) *$b=0$ and $c=-3a$. In this case, $f=a(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$, where $\omega=e^{2\pi i/3}$.*

(3) *$R=0$ and $3b-3a-c\neq 0$. In this case,*

$$f=\frac{1}{(3b-3a-c)^2}\big((c-2b)x+(b-3a)(y+z)\big)\cdot\big((c-2b)y+(b-3a)(x+z)\big)\big((c-2b)z+(b-3a)(x+y)\big).$$

(Note, if $R=0$ and $3b-3a-c=0$ then case(1) occurs.)

Proof. This an easy consequence of Theorem 3.5 and Corollary 3.7.

$\square$

Example 3.9. *Suppose we have the polynomial* $f = (x^3 + y^3 + z^3) + (x^2y + xy^2 + x^2z + y^2z + xz^2 + yz^2)$*. Since* $3b - 3a - c = 0$*, we have* $f = (x + y + z)(x^2 + y^2 + z^2)$ *as in Theorem* 3.5*.*

Example 3.10. *Suppose we have the polynomial* $f = (x^3 + y^3 + z^3) + 3(x^2y + xy^2 + x^2z + y^2z + xz^2 + yz^2) + 6xyz$*. As* $b = 3a$ *and* $c = 6a$ *where* $a = 1$*, then* $f = (x+y+z)^3$ *as in Theorem* 3.8*.*

Example 3.11. *Suppose we have the polynomial* $f = 4(x^3 + y^3 + z^3) - 12xyz$*, as* $b = 0$*,* $c = 3a$ *and* $a = 1$ *then,* $f = 4(x + y + z)(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$*, where* $\omega = e^{2\pi i/3}$ *as in Theorem* 3.8*.*

Corollary 3.12. *Let* $f = a(x^3+y^3+z^3)+b(x^2y+xy^2+x^2z+y^2z+xz^2+yz^2)+cxyz$ *for some* $a, b, c \in \mathbb{C}$*. Then* f *has the form*

$$f = (\alpha x + \beta(y + z))(\alpha y + \beta(x + z))(\alpha z + \beta(x + y))$$

where $\alpha, \beta \in \mathbb{C}$ *if and only if* $R = 0$*.*

Proof. This is an extremely easy consequence of Theorem 3.8(3). $\square$

CHAPTER 4

Factorization of Degree Four Symmetric Polynomials

In this chapter we will discuss the reducibility of degree four symmetric polynomials. There are essentially four different cases that can occur. The polynomial can factor into a degree one polynomial and a degree three polynomial, two degree one polynomials and a degree two polynomial, four degree one polynomials, or two degree two polynomials. Some of these cases can overlap. As the previous chapter, we commence by looking for a general form for a symmetric, trivariate, degree four polynomial.

Lemma 4.1. *A polynomial f is homogeneous, symmetric and has degree 4 if and only if it has the form:*

$$\begin{aligned} f = a(x^4+y^4+z^4) + b(x^2y^2+x^2z^2+y^2z^2) \\ + c(x^3y+x^3z+y^3x+y^3z+z^3x+z^3y) + d(x^2yz+y^2xz+z^2xy), \end{aligned} \tag{4.1}$$

where $a, b, c, d \in \mathbb{C}$.

Proof. If f is a homogeneous degree 4 polynomial, it can be written as

$$\begin{aligned} f = c_1x^4 + c_2y^4 + c_3z^4 + c_4x^2y^2 + c_5x^2z^2 + c_6y^2z^2 \\ + c_7x^3y + c_8x^3z + c_9y^3x + c_{10}y^3z + c_{11}z^3x \\ + c_{12}z^3y + c_{13}x^2yz + c_{14}y^2xz + c_{15}z^2xy, \end{aligned}$$

for some $c_1, c_2, \ldots, c_{15} \in \mathbb{C}$.

If, in addition, f is symmetric, then $(x\ y)\cdot f = (z\ y\ x)\cdot f = f$. Consider,

$$
\begin{aligned}
f = (x\ y)\cdot f = c_1y^4 &+ c_2x^4 + c_3z^4 + c_4x^2y^2 + c_5y^2z^2 + c_6x^2z^2 \\
&+ c_7y^3x + c_8y^3z + c_9x^3y + c_{10}x^3z + c_{11}z^3y \\
&+ c_{12}z^3x + c_{13}y^2xz + c_{14}x^2yz + c_{15}z^2xy.
\end{aligned}
$$

Also note that

$$
\begin{aligned}
f = (z\ y\ x)\cdot f = c_1y^4 &+ c_2z^4 + c_3x^4 + c_4y^2z^2 + c_5y^2x^2 + c_6z^2x^2 \\
&+ c_7y^3z + c_8y^3x + c_9z^3y + c_{10}z^3x + c_{11}x^3y \\
&+ c_{12}x^3z + c_{13}y^2zx + c_{14}z^2yx + c_{15}x^2yz.
\end{aligned}
$$

Matching coefficients we obtain:

$$
\begin{gathered}
c_1 = c_2 = c_3, \quad c_4 = c_5 = c_6, \\
c_7 = c_8 = c_9 = c_{10} = c_{11} = c_{12} \\
c_{13} = c_{14} = c_{15}.
\end{gathered}
$$

Consequently, by setting $a = c_1$, $b = c_4$, $c = c_7$, and $d = c_{13}$, f has the form of (4.1).

Conversely, if f has the form of (4.1), it's clear that f is a degree 4 symmetric homogeneous polynomial. $\square$

We shall now discuss the four different possible factorizations of degree four symmetric polynomial mentioned above. Our main tools are the unique factorization property of polynomials and some group theory.

Theorem 4.2. *Let f be a trivariate, homogeneous, degree 4 and symmetric polynomial. If $f = gh$ with $\deg g = 3$, $\deg h = 1$ and g irreducible, then $h = \alpha(x+y+z)$ for some $\alpha \in \mathbb{C}^{\times}$ and g is symmetric.*

Proof. Suppose $f = gh$, $\deg g = 3$, g is irreducible and $\deg h = 1$. By Corollary 1.13, g and h are symmetric. By Lemma 1.14, $h = \alpha(x+y+z)$ for some $\alpha \in \mathbb{C}^{\times}$. □

Theorem 4.3. *Let f be a trivariate, homogeneous, degree 4 and symmetric polynomial. If $f = ghk$ where $\deg h = \deg k = 1$ and $\deg g = 2$ with g irreducible, then*

(1) *g is symmetric*

(2) *$hk = a_1(x + \lambda y + \lambda^2 z)(x + \lambda^2 y + \lambda z)$ for some $a_1, \lambda \in \mathbb{C}^{\times}$ such that $\lambda^3 = 1$*

Proof. Suppose $f = ghk$ where $\deg h = \deg k = 1$ and $\deg g = 2$ where g is irreducible. Let $\sigma \in S_{\{x,y,z\}}$. By Corollary 1.13, g and hk are symmetric.

Since hk is symmetric and reducible, by Theorem 2.5, $hk = a_1(x + \lambda y + \lambda^2 z)(x + \lambda^2 y + \lambda z)$ for some $a_1, \lambda \in \mathbb{C}^{\times}$ and $\lambda^3 = 1$. □

Theorem 4.4. *Let f be a trivariate, homogeneous, degree 4 and symmetric polynomial. If f is a product of four degree one polynomials, then one of the following occurs:*

(1) *$f = a(x+\lambda y+\lambda^2 z)(x+\lambda^2 y+\lambda z)(x+\tau y+\tau^2 z)(x+\tau^2 y+\tau z)$ for some $\lambda, \tau \in \mathbb{C}$ where $\lambda^3 = 1$ and $\tau^3 = 1$.*

(2) *$f = \gamma(x+y+z)(c_1 x + b_1(y+z))(c_1 y + b_1(x+z))(c_1 z + b_1(x+y))$ for some $\gamma, b_1, c_1 \in \mathbb{C}^{\times}$ and $b_1 \neq c_1$.*

Proof. Let $f = ghkl$, where $\deg g = \deg h = \deg k = \deg l = 1$. By the unique factorization theorem of polynomials we get a group homomorphism $\phi : S_{\{x,y,z\}} \to S_T$

where $T = \{\mathbb{C}^\times g, \mathbb{C}^\times h, \mathbb{C}^\times k, \mathbb{C}^\times l\}$ as in Lemma 1.11. The proof now splits into three cases depending on the kernel of ϕ.

Case (1): Suppose that $\ker\phi = S_{\{x,y,z\}}$. This means that each element in $S_{\{x,y,z\}}$ gets sent to the identity element of S_T. In particular, $\sigma \cdot g = \lambda_\sigma g$ for all $\sigma \in S_{\{x,y,z\}}$ and, by Lemma 1.14, this implies that g is a scalar multiple of $x+y+z$. Similarly h, k and l are also scalar multiples of $x+y+z$. This implies that $f = a(x+y+z)^4$, a special case of (1) with $\lambda = \tau = 1$.

Case (2): Suppose that $\ker\phi = \{1, (z\ y\ x), (x\ y\ z)\}$. Then $|\operatorname{im}\phi| = 2$. This implies that $\operatorname{im}\phi = \{1_T, \mu\}$, where $\mu \in S_T$ and $|\mu| = 2$. Now, μ is either a transposition, or a product of two disjoint transpositions. Without loss of generality, $\operatorname{im}\phi$ is $\{1, (\mathbb{C}^\times g, \mathbb{C}^\times h)\}$ or $\{1, (\mathbb{C}^\times g, \mathbb{C}^\times h)(\mathbb{C}^\times k, \mathbb{C}^\times l)\}$.

Case (2a): Suppose that $\operatorname{im}\phi = \{1, (\mathbb{C}^\times g, \mathbb{C}^\times h)\}$. This implies that k and l are scalar multiples of $x+y+z$. Now, we have that $f = gh\alpha(x+y+z)^2$ where $\alpha \in \mathbb{C}^\times$. Since f is symmetric and $\alpha(x+y+z)^2$ is symmetric, Lemma 1.6 implies that gh is symmetric. Because gh is reducible, by Theorem 2.5, gh is a scalar multiple of $(x+\lambda y+\lambda^2 z)(x+\lambda^2 y+\lambda z)$, where $\lambda^3 = 1$. This implies that $f = a(x+y+z)^2(x+\lambda y+\lambda^2 z)(x+\lambda^2 y+\lambda z)$, which is equation (2) with $\tau = 1$.

Case (2b): Now, suppose that $\operatorname{im}\phi = \{1, (\mathbb{C}^\times g, \mathbb{C}^\times h)(\mathbb{C}^\times k, \mathbb{C}^\times l)\}$. As a result, gh and kl are each almost symmetric. By Theorem 1.9, the products gh and kl are symmetric as each product has degree 2. Then, by Theorem 2.5, $f = a(x+\lambda y+\lambda^2 z)(x+\lambda^2 y+\lambda z)(x+\tau y+\tau^2 z)(x+\tau^2 y+\tau z)$, where $\lambda^3 = 1$ and $\tau^3 = 1$.

Case (3): Suppose that $\ker\phi = \{1\}$. Then, we have $\phi(S_{\{x,y,z\}}) \cong S_3$ and so $|T| = 3$ or $|T| = 4$.

We will show by contradiction that $|T| = 3$ is not possible. If $|T| = 3$, then without loss of generality, $T = \{\mathbb{C}^\times g, \mathbb{C}^\times h, \mathbb{C}^\times k\}$ and $\mathbb{C}^\times k = \mathbb{C}^\times l$. Since the group $S_{\{x,y,z\}}$ permutes the elements of $T = \{\mathbb{C}^\times g, \mathbb{C}^\times h, \mathbb{C}^\times k\}$, ghk is fixed by all elements of the group $S_{\{x,y,z\}}$ up to scalar multiples, therefore, ghk is almost symmetric. Consider the following: $(ghk)l = f = \sigma \cdot f = (\sigma \cdot (ghk))(\sigma \cdot l) = (\lambda_\sigma ghk)(\sigma \cdot l)$. By cancelation, $\lambda_\sigma \sigma \cdot l = l$. Since σ was arbitrary, l is almost symmetric by Theorem 1.9 and hence symmetric by Corollary 1.10. Since l is symmetric, by Lemma 1.14, $l = \alpha_1(x + y + z)$ and $k = \alpha_2(x + y + z)$ as $\mathbb{C}^\times k = \mathbb{C}^\times l$ for some α_1, α_2. Since $\mathbb{C}^\times k \in T$, then without loss of generality, $\sigma \cdot k = \lambda_\sigma h$ for some $\sigma \in S_{\{x,y,z\}}$. Since k is a multiple of $x + y + z$, then so is h by construction, hence $\mathbb{C}^\times k = \mathbb{C}^\times l = \mathbb{C}^\times h$ and $|T| \leq 2$, contradicting the assumption that $|T| = 3$.

So $|T| = 4$ and $\{\mathbb{C}^\times g, \mathbb{C}^\times h, \mathbb{C}^\times k, \mathbb{C}^\times l\}$ are distinct elements of T. Without loss of generality, $\mathbb{C}^\times g$, $\mathbb{C}^\times h$ and $\mathbb{C}^\times k$ are permuted amongst themselves, and $\mathbb{C}^\times l$ is fixed by all elements of $S_{\{x,y,z\}}$. As $\mathbb{C}^\times l$ is fixed by all elements of $S_{\{x,y,z\}}$, l is almost symmetric. Because l has degree one, l is symmetric by Theorem 1.9 and l is a scalar multiple of $x + y + z$ by Lemma 1.14. Now, since l and f are symmetric, ghk is symmetric by Lemma 1.6.

The fact that $\mathbb{C}^\times g$, $\mathbb{C}^\times h$ and $\mathbb{C}^\times k$ are permuted amongst themselves implies that

g, h and k are not symmetric. Since ghk is a completely reducible symmetric degree three polynomial, by Theorem 3.8, ghk is a scalar multiple of

$$(c_1x + b_1(y+z))(b_1x + c_1y + b_1z)(b_1x + b_1y + c_1z)$$

for some $b_1, c_1 \in \mathbb{C}$, or a scalar multiple of

$$(x+y+z)(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$$

where $\omega = e^{2\pi i/3}$.

Therefore, f is a scalar multiple of

$$(x+y+z)(c_1x + b_1(y+z))(c_1y + b_1(x+z))(c_1z + b_1(x+y))$$

or a scalar multiple of

$$(x+y+z)^2(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z).$$

In the first case, f has the form of (2) and in the second case f has the form of (1) with $\tau = 1$ and $\lambda = \omega$. $\square$

Theorem 4.5. *Let f be a trivariate, homogeneous, degree 4 and symmetric polynomial. If $f = gh$ where $\deg g = \deg h = 2$ and g, h are irreducible, then g and h are symmetric or*

$$\begin{aligned} f = A(\alpha(x^2+\lambda^2y^2+\lambda z^2) + \beta(xy+\lambda^2yz+\lambda xz)) \\ \cdot (\alpha(y^2+\lambda^2x^2+\lambda z^2) + \beta(xy+\lambda^2xz+\lambda yz)) \end{aligned} \tag{4.2}$$

for some $\alpha, \beta, \lambda, A \in \mathbb{C}$ with $\lambda^3 = 1$.

Proof. By the unique factorization theorem of polynomials we get a group homomorphism $\phi : S_{\{x,y,z\}} \to S_T$ where $T = \{\mathbb{C}^\times g, \mathbb{C}^\times h\}$ as in Lemma 1.11. We consider the three different cases for the kernel of this homomorphism.

(1) Suppose $\ker\phi = S_{\{x,y,z\}}$. As a result, g and h are almost symmetric. However, since g and h are each degree 2, by Theorem 1.9, g and h are symmetric.

(2) Suppose $\ker\phi = \{(z\ y\ x), (x\ y\ z), 1\}$. As $(z\ y\ x) \in \ker\phi$, then, $(z\ y\ x)\cdot g = \lambda g$ for some $\lambda \in \mathbb{C}$ and, by Lemma 2.1, $\lambda^3 = 1$ and $g = \alpha(x^2+\lambda^2y^2+\lambda z^2)+\beta(xy+\lambda^2yz+\lambda xz)$ for some $\alpha, \beta \in \mathbb{C}$. Since $(x\ y) \notin \ker\phi$, h is a scalar multiple of $(x\ y)\cdot g$. That is,

$$\begin{aligned} h &= A(x\ y)\cdot(\alpha(x^2+\lambda^2y^2+\lambda z^2)+\beta(xy+\lambda^2yz+\lambda xz)) \\ &= A(\alpha(y^2+\lambda^2x^2+\lambda z^2)+\beta(xy+\lambda^2xz+\lambda yz)) \end{aligned}$$

for some $A \in \mathbb{C}$. Therefore, $f = gh$ has the claimed form.

(3) Suppose $\ker\phi = \{1\}$. This case cannot occur because, if it did, $\operatorname{im}\phi$ would be isomorphic to $S_{\{x,y,z\}}/(\ker\phi)$, a group of order 6. This is clearly impossible as $\operatorname{im}\phi \le S_T$ where $|S_T| \le 2$. □

We now know all possible factorizations of symmetric trivariate homogeneous degree four polynomials. Our next goal is to find conditions on the coefficients of these polynomials that determine which factorizations occur.

Theorem 4.6. *Let f be a trivariate, homogeneous, degree 4 and symmetric polynomial as in (4.1). Then $(x+y+z) \mid f$ if and only if $2a+b-2c=0$. In this* *circumstance,*

$$f = (x+y+z)\Big(a(x^3+y^3+z^3) \\ + (c-a)(xy^2+xz^2+yx^2+yz^2+zx^2+zy^2) + (2a-2c+d)xyz\Big). \tag{4.3}$$

Proof. Suppose $(x+y+z) \mid f$, that is, $f(x,y,z) = (x+y+z)g(x,y,z)$. Plugging in $x = 1, y = -1, z = 0$ into f we get, $f(1,-1,0) = (1-1+0)g(1,-1,0) = 0$. Plugging in the same values into the equation $f = a(x^4+y^4+z^4)+b(x^2y^2+x^2z^2+y^2z^2)+c(x^3y+x^3z+y^3x+y^3z+z^3x+z^3y)+d(x^2yz+y^2xz+z^2xy)$ we get $f(1,-1,0) = 2a+b-2c$. Comparing this with the previous value of $f(1,-1,0) = 0$, we have $2a+b-2c = 0$.

Conversely, suppose $2a+b-2c = 0$. This implies that $b = -2(a-c)$. Plugging in this condition into (4.1) we get, after factoring, (4.3). This makes it clear that $(x+y+z) \mid f$. □

Example 4.7. *Suppose*

$$f = 10(x^4+y^4+z^4) \\ + 10(x^3y+x^3z+y^3x+y^3z+z^3x+z^3y) + 1000(x^2yz+y^2xz+z^2xy), \tag{4.4}$$

with $a = c = 10$*,* $b = 0$ *and* $d = 1000$*. Since* $2a+b-2c = 0$ *holds, we can apply Theorem* 4.6 *to factor* f*:*

$$f = 10(x+y+z)(x^3+y^3+z^3+100xyz).$$

It is interesting that the reducibility of f *does not depend on* d *at all. This is because the term of* f *containing* d *is already a multiple of* $x+y+z$*. Also, the factorization in Theorem* 4.6 *can hold for even a bizarre choice of* a*,* b *and* c *— as long as the equation* $2a+b-2c = 0$ *is satisfied.*

Theorem 4.8. *Let f be a trivariate, homogeneous, degree 4 and symmetric polynomial as in* (4.1)*. Then, f has the form*

$$\gamma(x+y+z)(c_1x+b_1(y+z))(c_1y+b_1(x+z))(c_1z+b_1(x+y))$$

for some $\gamma, b_1, c_1 \in \mathbb{C}^\times$ and $b_1 \neq c_1$ if and only if $2a+b-2c=0$, $-8a+5c-d \neq 0$, and $16a^2c-11ac^2+4c^3-2acd-c^2d+ad^2=0$. In this case,

$$f = \frac{-1}{(8a-5c+d)^2}(x+y+z)\big((4a-c)(x+z)+(4c-4a-d)y\big)$$
$$\cdot\big((4a-c)(x+y)+(4c-4a-d)z\big)\big((4a-c)(y+z)+(4c-4a-d)x\big).$$

Proof. By Theorem 4.6, $x+y+z$ is a factor of f if and only if $2a+b-2c=0$. When this happens, f factors as in (4.3). The rest of the claim follows from applying Theorem 3.5(2) to the degree 3 factor of f in (4.3). □

Theorem 4.9. *Let f be a trivariate, homogeneous, degree 4 and symmetric polynomial as in* (4.1)*. Then f has the form*

$$\left(\beta(xy+xz+yz)+\alpha(x^2+y^2+z^2)\right)\left(\delta(xy+xz+yz)+\gamma(x^2+y^2+z^2)\right),$$

for some $\alpha,\beta,\gamma,\delta \in \mathbb{C}$ if and only if $2b-4a+c-d=0$. When this happens α,β,γ and δ are determined by a factorization $ax^2+cxy+(b-2a)y^2=(\alpha x+\beta y)(\gamma x+\delta y)$.

Proof. By matching coefficients, f has the claimed form if and only if

$$a=\alpha\gamma \qquad b=2\alpha\gamma+\beta\delta \qquad c=\beta\gamma+\alpha\delta \qquad d=\beta\gamma+\alpha\delta+2\beta\delta \tag{4.5}$$

If f factors as claimed, then plugging in the equations in (4.5) into $2b-4a+c-d$ we get that $2b-4a+c-d=0$.

Conversely, suppose $2b - 4a + c - d = 0$. Consider the polynomial

$$Q = ax^2 + cxy + (b - 2a)y^2.$$

Since Q is bivariate, it factors over $\mathbb{C}$. Let $\alpha, \beta, \gamma, \delta$ be determined by such a factorization: $Q = (\alpha x + \beta y)(\gamma x + \delta y)$. By matching coefficients in this equation we obtain $\alpha\gamma = a$, $\alpha\delta + \beta\gamma = c$, and $b = 2\alpha\gamma + \beta\delta$. Plugging in these equations for a, b, c into $d = 2b - 4a + c$, we find that $d = \beta\gamma + \alpha\delta + 2\beta\delta$. Since the equations (4.5) above are satisfied, f has the desired factorization. $\square$

Theorem 4.10. *Let f be a trivariate, homogeneous, degree 4 and symmetric polynomial as in (4.1). Set*

$$S = a + b + d + 2c \quad T = a^2 + ab - c^2 \quad U = 2ac + c^2 + ad.$$

Then f has the form

$$f = A\left(\alpha(\omega x^2 + y^2 + \omega^2 z^2) + \beta(\omega^2 xy + xz + \omega yz)\right)$$
$$\times \left(\alpha(\omega^2 x^2 + y^2 + \omega z^2) + \beta(\omega xy + xz + \omega^2 yz)\right)$$

with $A, \alpha, \beta \in \mathbb{C}$ and $\omega = e^{2\pi i/3}$ if and only if $S = T = U = 0$. Since $aS = U + T$, if $a \neq 0$, this implies that if two of these quantities are zero then so is the third.

Proof. If f has the claimed form, then matching coefficients we get

$$a = A\alpha^2 \qquad b = A(\beta^2 - \alpha^2) \qquad c = -A\alpha\beta \qquad d = A(2\alpha - \beta)\beta.$$

Plugging these expressions into the definitions of S, T and U gives $S = T = U = 0$.

The converse splits into two cases:

(1) Suppose that $S = T = U = 0$ and $a \neq 0$. Set

$$F = \big(a(\omega x^2 + y^2 + \omega^2 z^2) - c(\omega^2 xy + xz + \omega yz)\big)$$
$$\cdot \big(a(\omega^2 x^2 + y^2 + \omega z^2) - c(\omega xy + xz + \omega^2 yz)\big).$$

Then a tedious calculation, done without the assumption $S = T = U = 0$, gives

$$af = F + (x^2y^2 + x^2z^2 + y^2z^2)T + (x^2yz + xy^2z + xyz^2)U.$$

Since $U = T = 0$ and $a \neq 0$, we have $f = \frac{1}{a}F$ and so f has the claimed form with $A = 1/a$, $\alpha = a$ and $\beta = -c$.

(2) Suppose that $S = T = U = 0$ and $a = 0$. These equations imply that $c = 0$ and $b + d = 0$, and so

$$f = b(x^2y^2 - x^2yz - xy^2z + x^2z^2 - xyz^2 + y^2z^2)$$
$$= b(\omega^2 xy + xz + \omega yz)(\omega xy + xz + \omega^2 yz).$$

Thus f has the claimed form with $A = b$, $\alpha = 0$ and $\beta = 1$. Note that the factorization above occurs if and only if $a = c = 0$ and $b = -d \neq 0$. □

Now we know all possible factorizations of trivariate, homogeneous, degree 4 and symmetric polynomials and we know conditions on the coefficients of f that determine when such factorizations occur. Next we can combine all this information into the main theorem of this chapter.

Theorem 4.11. *Let f be a trivariate, homogeneous, degree 4 and symmetric polynomial. Then, f is reducible if and only if at least one of the following occurs:*

(1) $2a+b-2c=0$. *In this case,*

$$f=(x+y+z)\Big(a(x^3+y^3+z^3)$$
$$+(c-a)(xy^2+xz^2+yx^2+yz^2+zx^2+zy^2)+(2a-2c+d)xyz\Big).$$

(a) *In addition, if* $-8a+5c-d=0$ *then,*

$$f=(x+y+z)^2((-2a+c)(xy+xz+yz)+a(x^2+y^2+z^2)).$$

i. *In addition, if* $c=4a$ *then,* $f=a(x+y+z)^4$.

ii. *In addition, if* $c=a$ *then,* $f=a(x+y+z)^2(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$.

(b) *In addition, if* $-8a+5c-d\neq 0$ *and* $16a^2c-11ac^2+4c^3-2acd-c^2d+ad^2=0$,

then

$$f=\frac{-1}{(8a-5c+d)^2}(x+y+z)\big((4a-c)(x+z)+(4c-4a-d)y\big)$$
$$\cdot\big((4a-c)(x+y)+(4c-4a-d)z\big)\big((4a-c)(y+z)+(4c-4a-d)x\big)$$

(2) $2b-4a+c-d=0$. *In this case,*

$$f=\big(\beta(xy+xz+yz)+\alpha(x^2+y^2+z^2)\big)$$
$$\cdot\big(\delta(xy+xz+yz)+\gamma(x^2+y^2+z^2)\big)$$

for some $\alpha,\beta,\gamma,\delta\in\mathbb{C}$, *that are determined by the factorization*

$$ax^2+cxy+(b-2a)y^2=(\alpha x+\beta y)(\gamma x+\delta y). \tag{4.6}$$

In addition, at least one of the following occurs given it meets certain criteria:

(a) *If* $b = 6a$ *and* $c = 4a$ *then,* $f = a(x+y+z)^4$

(b) *If* $b = 3a$ *and* $c = -2a$ *then,* $f = a(x+\omega y+\omega^2 z)^2(x+\omega^2 y+\omega z)^2$

(c) *If* $b = 0$ *and* $c = a$ *then,* $f = a(x+y+z)^2(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$

(3) $a = c = 0$*, and* $b = -d \neq 0$*. In this case,*

$$f = b(\omega^2 xy + xz + \omega yz)(\omega xy + xz + \omega^2 yz).$$

(4) $a \neq 0$*,* $S = a+b+d+2c = 0$*,* $T = a^2+ab-c^2 = 0$ *and* $U = 2ac+c^2+ad = 0$*.*

In this case, f *has the form*

$$f = \frac{1}{a}\left(a(\omega x^2 + y^2 + \omega^2 z^2) - c(\omega^2 xy + xz + \omega yz)\right)$$
$$\cdot \left(a(\omega^2 x^2 + y^2 + \omega z^2) - c(\omega xy + xz + \omega^2 yz)\right).$$

In all of these cases, $\omega = e^{2\pi i/3}$*.*

Proof. Since $\deg f = 4$, f must factor into one degree one polynomial and a degree three polynomial, four degree one polynomials, two degree one polynomials and one degree two polynomials, or two degree two polynomials (assuming the each polynomial in the factorization of f is irreducible).

(1) Suppose that $f = gh$ factors into a degree one polynomial h and a degree three irreducible polynomial g. By Theorem 4.2, h is a multiple of $x+y+z$ and f factors as in case (1). (Since g is irreducible, f does not factor as in case (1a) or (1b).)

(2) Suppose that f factors as a product of four degree one polynomials. Using Theorem 4.4, our argument splits up into two cases:

(a) If f factors as in Theorem 4.4(1), it factors as in (2a), (2b) or (2c).

(b) If f factors as in Theorem 4.4(2), then Theorem 4.8 implies that f factors as in case (1b) .

(3) Suppose that f factors into two degree one polynomials and a degree two polynomial. By Theorem 4.3, f is a product of two degree two symmetric polynomials. By Theorem 4.9, $2b-4a+c-d=0$ and f factors as item (2), where one factor is reducible and the other one is not.

(4) Suppose that $f=gh$ is a product of two degree two irreducible polynomials. Now, by Theorem 4.5, we have 2 cases to consider;

(a) g and h are symmetric. By Theorem 4.9, $2b-4a+c-d=0$ and f factors as case (2).

(b) g and h are not symmetric and $a \neq 0$. By Theorem 4.5, f has the form of (4.2). By Theorem 4.10, $S=a+b+d+2c=0$, $T=a^2+ab-c^2=0$ and $U=2ac+c^2+ad=0$ and f factors as in case (4). □

In the next two examples, $2a+b-2c=0$, hence we can apply Theorem 4.6 to factor them.

Example 4.12. *Let*

$$f = 10(x^4+y^4+z^4)+10(x^3y+x^3z+y^3x+y^3z+z^3x+z^3y)$$
$$+974567(x^2yz+y^2xz+z^2xy).$$

Then, $a=c=10$, $b=0$ and $d=974567$, so case (1) of Theorem 4.11 applies and

$$f=(x+y+z)(10x^3+10y^3+10z^3+974567xyz)$$

.

Example 4.13. *Let* $a = 10$*,* $b = 2$*,* $c = 11$ *and* $d = \pi\sqrt{5017}$*, so case (1) of Theorem* 4.11 *applies and*

$$f = 10(x^4 + y^4 + z^4) + 2(x^2y^2 + x^2z^2 + y^2z^2)$$
$$+ 11(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + \pi\sqrt{5017}(x^2yz + y^2xz + z^2xy)$$
$$= (x + y + z)$$
$$\cdot (10x^3 + 10y^3 + 10z^3 + x^2y + xy^2 + x^2z + y^2z + xz^2 + yz^2 + (\pi\sqrt{5017} - 2)\, xyz).$$

Example 4.14. *If* $a = 1$*,* $b = 6$*,* $c = 4$ *and* $d = 12$*, then case (1ai) of Theorem* 4.11 *holds and we get:*

$$f = 1(x^4 + y^4 + z^4) + 6(x^2y^2 + x^2z^2 + y^2z^2)$$
$$+ 4(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + 12(x^2yz + y^2xz + z^2xy)$$
$$= (x + y + z)^4.$$

Example 4.15. *If* $a = 2$*,* $b = 6$*,* $c = 4$*, and* $d = 8$*, then case (3) of Theorem* 4.11 *holds and we get:*

$$f = 2(x^4 + y^4 + z^4) + 6(x^2y^2 + x^2z^2 + y^2z^2)$$
$$+ 4(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + 8(x^2yz + y^2xz + z^2xy)$$
$$= 2(x^2 + xy + y^2 + xz + yz + z^2)^2.$$

Example 4.16. *Let* $a = 1$*,* $b = 4$*,* $c = 10$ *and* $d = 14$*. Since* $2b - 4a + c - d = 0$*,* f *factors as in case (2) of Theorem* 4.11*. To determine* $\alpha, \beta, \gamma, \delta$ *we need the factorization*

$$x^2 + 10xy + (4 - 2)y^2 = (x + (5 + \sqrt{23})y)(x + (5 - \sqrt{23})y).$$

We choose $\alpha = \gamma = 1$, $\beta = 5 + \sqrt{23}$ *and* $\delta = 5 - \sqrt{23}$, *therefore,*

$$\begin{aligned} f &= (x^4 + y^4 + z^4) + 4(x^2y^2 + x^2z^2 + y^2z^2) \\ &\quad + 10(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + 14(x^2yz + y^2xz + z^2xy) \\ &= \Big((5 + \sqrt{23})(xy + xz + yz) + (x^2 + y^2 + z^2)\Big) \\ &\quad \cdot \Big((5 - \sqrt{23})(xy + xz + yz) + (x^2 + y^2 + z^2)\Big) . \end{aligned}$$

Example 4.17. *Let* $a = c = 0$ *and* $b = -d = 1$. *Since* $a + b + d + 2c = 0$, $a^2 + ab - c^2 = 0$, *and* $2ac + c^2 + ad = 0$, f *factors as in case (4) of Theorem* 4.11.

$$\begin{aligned} f &= (x^2y^2 + x^2z^2 + y^2z^2) - (x^2yz + y^2xz + z^2xy) \\ &= (\omega^2xy + xz + \omega yz)(\omega xy + xz + \omega^2yz). \end{aligned}$$

Example 4.18. *Let* $a = -c = d = 1$ *and* $b = 0$. *Since* $S = a + b + d + 2c = 0$, $T = a^2 + ab - c^2 = 0$ *and* $U = 2ac + c^2 + ad = 0$, f *factors as in case (5) of Theorem* 4.11.

$$\begin{aligned} f &= (x^4 + y^4 + z^4) \\ &\quad - (x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + (x^2yz + y^2xz + z^2xy) \\ &= \big((\omega x^2 + y^2 + \omega^2z^2) + (\omega^2xy + xz + \omega yz)\big) \\ &\quad \cdot \big((\omega^2x^2 + y^2 + \omega z^2) + (\omega xy + xz + \omega^2yz)\big) . \end{aligned}$$

CHAPTER 5

Factorization of Degree Five Symmetric Polynomials

In this chapter, we will discuss the factorization of degree five homogeneous symmetric polynomials. A degree five polynomial can factor in various ways: five degree one polynomials, a degree four polynomial and a degree one polynomial, a degree three polynomial and a degree two polynomial, a degree three polynomial and two degree one polynomials, two degree two polynomials and a degree one polynomial, and a degree three polynomial and two degree one polynomials.

Lemma 5.1. *A polynomial f is homogeneous, symmetric and has degree 5 if and only if it has the form:*

$$\begin{aligned} f = {} & a(x^5+y^5+z^5) + b(x^4y+x^4z+y^4x+y^4z+z^4x+z^4y) \\ & + c(x^3y^2+x^3z^2+y^3x^2+y^3z^2+z^3x^2+z^3y^2) \\ & + d(x^2y^2z+z^2y^2x+z^2x^2y) + e(x^3yz+y^3xz+z^3xy) \end{aligned} \tag{5.1}$$

where $a, b, c, d, e \in \mathbb{C}$.

Proof. If f is a homogeneous degree 5 polynomial, it can be written as

$$\begin{aligned} f = {} & c_1x^5 + c_2y^5 + c_3z^5 + c_4x^4y + c_5x^4z + c_6y^4x + c_7y^4z + c_8z^4x + c_9z^4y \\ & + c_{10}x^3y^2 + c_{11}x^3z^2 + c_{12}y^3x^2 + c_{13}y^3z^2 + c_{14}z^3x^2 + c_{15}z^3y^2 \\ & + c_{16}x^2y^2z + c_{17}z^2y^2x + c_{18}z^2x^2y + c_{19}x^3yz + c_{20}y^3xz + c_{21}z^3xy \end{aligned} \tag{5.2}$$

Since f is a symmetric polynomial, $\sigma \cdot f = f$ for all $\sigma \in S_{\{x,y,z\}}$. As in previous theorems, the fact that $(x\ y) \cdot f = (x\ y\ z) \cdot f = f$ implies that f has the claimed form. The converse is trivial. $\square$

Theorem 5.2. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial. If f is a product of five degree one polynomials, then one of the following occurs:*

(A) $f = \mu(x+y+z)^2(A(x+y)+Bz)(A(x+z)+By)(A(z+y)+Bx)$ *for some* $\mu, A, B, C \in \mathbb{C}$.

(B) $f = M(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)(A(x+y)+Bz)(A(x+z)+By)(A(z+y)+Bx)$ *for some* $M, A, B, C \in \mathbb{C}$.

(C) $f = a(x+y+z)(x+\lambda y+\lambda^2 z)(x+\lambda^2 y+\lambda z)(x+\tau y+\tau^2 z)(x+\tau^2 y+\tau z)$, *for some* $\lambda^3 = 1$ *and* $\tau^3 = 1$.

Proof. Let $f = ghklm$, where $\deg g = \deg h = \deg k = \deg l = \deg m = 1$. By the unique factorization theorem of polynomials we get a group homomorphism $\phi : S_{\{x,y,z\}} \to S_T$ where $T = \{\mathbb{C}^\times g, \mathbb{C}^\times h, \mathbb{C}^\times k, \mathbb{C}^\times l, \mathbb{C}^\times m\}$ as in Lemma 1.11. The proof now splits into three cases depending on the kernel of ϕ.

Case (1): Suppose that $\ker\phi = S_{\{x,y,z\}}$. This means that each element in $S_{\{x,y,z\}}$ gets sent to the identity element of S_T. In particular, $\sigma \cdot g = \lambda_\sigma g$ for all $\sigma \in S_{\{x,y,z\}}$ and, by Lemma 1.14, this implies that g is a scalar multiple of $x+y+z$. Similarly h, k, l and m are also scalar multiples of $x+y+z$. This implies that $f = a(x+y+z)^5$ and this is a special case of (A) and (C).

Case (2): Suppose that $\ker\phi = \{1, (z\ y\ x), (x\ y\ z)\}$. Then $|\operatorname{im}\phi| = 2$. This implies that $\operatorname{im}\phi = \{1_T, \mu\}$, where $\mu \in S_T$ and $|\mu| = 2$. Now, μ is either a transposition, or a product of two disjoint transpositions. Without loss of generality, $\operatorname{im}\phi$ is $\{1, (\mathbb{C}^\times g, \mathbb{C}^\times h)\}$ or $\{1, (\mathbb{C}^\times g, \mathbb{C}^\times h)(\mathbb{C}^\times k, \mathbb{C}^\times l)\}$.

Case (2a): Suppose that $\operatorname{im}\phi = \{1, (\mathbb{C}^\times g, \mathbb{C}^\times h)\}$. This implies that k, l and m are scalar multiples of $x+y+z$. Now, we have that $f = gh\alpha(x+y+z)^3$ where $\alpha \in \mathbb{C}^\times$. Since f is symmetric and $\alpha(x+y+z)^3$ is symmetric, Lemma 1.6 implies that gh is symmetric. Because gh is reducible, by Theorem 2.5, gh is a scalar multiple of $(x+\lambda y+\lambda^2 z)(x+\lambda^2 y+\lambda z)$, where $\lambda^3 = 1$. This implies that $f = a(x+y+z)^3(x+\lambda y+\lambda^2 z)(x+\lambda^2 y+\lambda z)$. This is a special case of (C) where $\tau = 1$.

Case (2b): Now, suppose that $\operatorname{im}\phi = \{1, (\mathbb{C}^\times g, \mathbb{C}^\times h)(\mathbb{C}^\times k, \mathbb{C}^\times l)\}$. As a result, m, gh and kl are each almost symmetric. By Theorem 1.9, the products m, gh and kl are symmetric as each product has degree less than 3. Then, by Theorem 2.5, $f = a(x+y+z)(x+\lambda y+\lambda^2 z)(x+\lambda^2 y+\lambda z)(x+\tau y+\tau^2 z)(x+\tau^2 y+\tau z)$, where $\lambda^3 = 1$ and $\tau^3 = 1$, as in (C).

Case (3): Suppose that $\ker\phi = \{1\}$. Then the image of the homomorphism ϕ is a subgroup of S_T that is isomorphic to S_3. Hence $|T| \leq 3$. Up to renumbering, the only subgroup of S_4 that is isomorphic to S_3 is S_3 itself. And, up to renumbering, the only subgroups of S_5 that are isomorphic to S_3 are S_3 and the subgroup generated by $(1\ 2)(4\ 5)$ and $(1\ 2\ 3)$, namely,

$$\langle (1\ 2)(4\ 5), (1\ 2\ 3)\rangle = \{1, (1\ 2\ 3), (3\ 2\ 1), (2\ 3)(4\ 5), (1\ 2)(4\ 5), (1\ 3)(4\ 5)\}.$$

In all of these cases, there is a 3 element subset of T, $U = \{\mathbb{C}^\times g, \mathbb{C}^\times h, \mathbb{C}^\times k\}$ say, such that $S_{\{x,y,z\}}$ permutes the elements of U amongst themselves. In particular, ghk is almost symmetric. In fact, ghk must be symmetric since $f = (g\ h\ k)(l\ m)$

implies lm is almost symmetric and since $\deg lm = 2$, lm is symmetric.

Now we have $f = (ghk)(lm)$ with ghk and lm symmetric and reducible. The possible forms of ghk are in Corollary 3.7 and the possible forms of lm are in Theorem 2.5. Combining these we see that f has the form (A), (B), or (C). □

Theorem 5.3. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial. If $f = ghk$ where $\deg g = \deg h = 2$ and $\deg k = 1$ and g, h and k are irreducible, then g, h and k are symmetric or*

$$\begin{aligned} f = A(x+y+z)(\alpha(x^2+\lambda^2 y^2+\lambda z^2)+\beta(xy+\lambda^2 yz+\lambda xz)) \\ \cdot(\alpha(y^2+\lambda^2 x^2+\lambda z^2)+\beta(xy+\lambda^2 xz+\lambda yz)) \end{aligned} \tag{5.3}$$

for some α, β, $A, \lambda \in \mathbb{C}$ with $\lambda^3 = 1$.

Proof. By Corollary 1.13, gh and k are symmetric. Since g and h are irreducible and $\deg g = \deg h = 2$, it follows from Theorem 4.5 that g and h are symmetric or gh is a scalar multiple of

$$\begin{aligned} (\alpha(x^2+\lambda^2 y^2+\lambda z^2)+\beta(xy+\lambda^2 yz+\lambda xz)) \\ \cdot(\alpha(y^2+\lambda^2 x^2+\lambda z^2)+\beta(xy+\lambda^2 xz+\lambda yz)), \end{aligned} \tag{5.4}$$

for some $\alpha, \beta, \lambda \in \mathbb{C}$ where $\lambda^3 = 1$. Hence, g, h and k are symmetric or $f = ghk$ has the form of (5.3) as k is a symmetric degree one polynomial, then k is a scalar multiple of $x+y+z$ by Lemma 1.14. □

Theorem 5.4. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial. If $f = ghk$ where $\deg g = \deg h = 1$ and $\deg k = 3$ and g, h and k are* *irreducible, then gh and k are symmetric and*

$$f = (x + \lambda y + \lambda^2 z)(x + \lambda^2 y + \lambda z)$$
$$\cdot (a_1(x^3 + y^3 + z^3) + b_1(xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2) + c_1 xyz)$$

for some $a_1, b_1, c_1, \lambda \in \mathbb{C}$ where $\lambda^3 = 1$.

Proof. By Corollary 1.13, gh and k are symmetric. As gh is symmetric, gh is a scalar multiple of $(x + \lambda y + \lambda^2 z)(x + \lambda^2 y + \lambda z)$ for some $\lambda \in \mathbb{C}$ and $\lambda^3 = 1$ as stated in Theorem 2.5. Because k is a symmetric degree 3 polynomial, $k = a_1(x^3 + y^3 + z^3) + b_1(xy^2 + xz^2 + yx^2 + yz^2 + zx^2 + zy^2) + c_1 xyz$ for some $a_1, b_1, c_1 \in \mathbb{C}$ by Lemma 3.1.

Hence $f = (x+\lambda y+\lambda^2 z)(x+\lambda^2 y+\lambda z)(a_1(x^3+y^3+z^3)+b_1(xy^2+xz^2+yx^2+yz^2 + zx^2 + zy^2) + c_1 xyz)$, for some $a_1, b_1, c_1, \lambda \in \mathbb{C}$ where $\lambda^3 = 1$. □

Theorem 5.5. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial.*

(1) *If $f = gh$ where $\deg g \neq \deg h$ and g and h are irreducible, then g and h are symmetric.*

(2) *If $f = gh$ where $\deg g = 3$ and g is irreducible, then g and h are symmetric.*

Proof. (1) Since f factors as in Corollary 1.13, g and h are symmetric.

(2) Either h is irreducible (in which case the result follows from (1)) or h is a product of two degree one polynomials (where each degree one polynomial need not be symmetric). In the second case, the claim is a direct consequence of Corollary 1.13. □

Theorem 5.6. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial. Suppose $f = ghkl$ with $\deg g = 2$, $\deg h = \deg k = \deg l = 1$ and g irreducible.*

Then g and hkl are symmetric and hkl has one of three forms from Theorem 3.8.

Proof. Since f factors as in Corollary 1.13, g and hkl are symmetric. By construction, hkl is completely reducible and thus has one of three forms from Theorem 3.8. □

We have seen proofs of the theoretical type. We are also interested in the conditions the coefficients have to have in order for particular factorizations to occur. That is precisely what these next theorems are all about.

Theorem 5.7. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial as in* (5.1)*. Set $W = 5a - 5b + c - d + 2e$, $U = 5a - 3b + c$ and $V = 15a - 7b + e$.*

(1) *$(x + y + z) \mid f$ if and only if $W = 0$.*

(2) *$(x + y + z)^2 \mid f$ if and only if $W = 0$ and $U = 0$.*

(3) *$(x + y + z)^3 \mid f$ if and only if $W = 0$, $U = 0$ and $V = 0$.*

(4) *$(x + y + z)^4 \mid f$ if and only if $(x + y + z)^5 \mid f$ if and only if $b = 5a$, $c = 10a$, $d = 30a$, and $e = 20a$.*

Proof. (1) Suppose $(x+y+z) \mid f$, that is, $f = (x+y+z)g$ for some $g \in \mathbb{C}[x, y, z]$. Plugging in $x = 2, y = -1, z = -1$ into f we get, $f(2, -1, -1) = (2-1-1)g = 0$. Plugging in the same values into (5.1), we get $f(2, -1, -1) = 5a-5b+c-d+2e$. Comparing this with the previous value of $f(2, -1, -1) = 0$ we have $W = 0$.

Conversely, suppose $W = 0$. The remainder of (5.1) divided by $x + y + z$ is a polynomial that, after factoring, has the coefficient $-(5a - 5b + c - d + 2e)$. As $5a - 5b + c - d + 2e = 0$, the remainder is 0 and hence $(x + y + z) \mid f$.

(2) Suppose $(x+y+z)^2 \mid f$, that is $f = (x+y+z)^2g$ for some $g \in \mathbb{C}[x, y, z]$. From (1) it is clear that $W = 0$. Equating coefficients of f in the form of (5.1) with

$f = (x+y+z)^2 g$ where g is a degree three polynomial of the form in (3.1) having coefficients A, B, and C, one gets that $a = A$, $b = 2A+B$, $c = A+3B$, $d = 6B+2C$, and $e = 2A+4B+C$. These relationships imply that $U = 0$.

Conversely, suppose $W = 0$ and $U = 0$. These equations imply that $d = 5a - 5b + c + 2e$ and $c = 3b - 5a$. Plugging in these conditions into f we find that f factors as

$$(x+y+z)^2 \big[a(x^3+y^3+z^3) + (b-2a)(x^2y + x^2z + y^2x + y^2z + z^2x + z^2y) \\ + (6a - 4b + e)xyz\big]. \tag{5.5}$$

Clearly $(x+y+z)^2 \mid f$.

(3) Suppose $(x+y+z)^3 \mid f$, that is $f = (x+y+z)^3 g$ for some $g \in \mathbb{C}[x,y,z]$. It is clear that the argument above will hold and that we'll get that $W = 0$ and $U = 0$. Equating coefficients of f in the form of (5.1) with $f = (x+y+z)^3 g$ where g is a degree two symmetric homogeneous polynomial of the form in (2.1) having coefficients A and B one gets that $a = A$, $b = 3A+B$, $c = 4A+3B$, $d = 6A+12B$, $e = 6A+7B$. These conditions show that $V = 0$.

Conversely, suppose $W = U = V = 0$. These equations imply that $d = 5a - 5b + c + 2e$, $c = 3b - 5a$ and $e = 7b - 15a$. Plugging in these equations into f we find that f factors as

$$(x+y+z)^3 (a(x^2+y^2+z^2) + (b-3a)(xy+xz+yz)).$$

Clearly $(x+y+z)^3 \mid f$.

(4) Suppose $f = (x+y+z)^4 g$ for some $g \in \mathbb{C}[x,y,z]$. As f and $(x+y+z)^4$ are symmetric, then by Lemma 1.6, g is symmetric. So, $g = \epsilon(x+y+z)$ for some $\epsilon \in \mathbb{C}$ by Lemma 1.14(2). So, $f = (x+y+z)^4 g = \epsilon(x+y+z)^5$. So, clearly $(x+y+z)^4 \mid f$ if and only if $(x+y+z)^5 \mid f$.

Next, by expanding $a(x+y+z)^5$ and equating coefficients with f as in (5.1), we see that $f = a(x+y+z)^5$ if and only if $b = 5a$, $c = 10a$, $d = 30a$, and $e = 20a$. $\square$

Let $L = (a-b)^2 + ae$ and $K = 3a - 4b - d + e$.

Theorem 5.8. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial as in (5.1). Then f has the form*

$$\begin{aligned} f = (x+y+z)&\left(\alpha(\omega x^2 + y^2 + \omega^2 z^2) + \beta(\omega^2 xy + xz + \omega yz)\right) \\ &\times \left(\alpha(\omega^2 x^2 + y^2 + \omega z^2) + \beta(\omega xy + xz + \omega^2 yz)\right) \end{aligned} \tag{5.6}$$

for some $\alpha, \beta \in \mathbb{C}$ and $\omega = e^{2\pi i/3}$ if and only if $W = K = L = 0$.

Proof. By equating coefficients of f in (5.6) with f as in (5.1), we see that f has the claimed form if and only if $a = \alpha^2$, $b = \alpha(\alpha - \beta)$, $c = -\alpha^2 - \alpha\beta + \beta^2$, $d = -\alpha^2 + 4\alpha\beta - \beta^2$, and $e = -\beta^2$. Plugging in these conditions into W, K, and L will result in 0.

Conversely, suppose $W = K = L = 0$. Because $L = 0$, we have $(a-b)^2 = -ae$. Therefore it is possible to choose α and β such that $\alpha^2 = a$, $\beta^2 = -e$ and $\alpha\beta = a - b$. These equations imply $b = \alpha(\alpha - \beta)$. The equation $K = 0$ now implies that $d = -\alpha^2 + 4\alpha\beta - \beta^2$ and similarly $W = 0$ implies $c = -\alpha^2 - \alpha\beta + \beta^2$. Therefore, f factors as claimed. $\square$

Let $M = 25a^3 - 40a^2b + 21ab^2 - 6b^3 - 5a^2e + 4abe + b^2e - ae^2$

Theorem 5.9. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial as in (5.1). Then f has the form*

$$(x+y+z)^2(c_1x + b_1(y+z))(c_1y + b_1(x+z))(c_1z + b_1(x+y))$$

for some $b_1, c_1 \in \mathbb{C}$ if and only if $W = U = M = 0$.

Proof. By Theorem 5.7, $(x+y+z)^2$ is a factor of f if and only if $W = U = 0$ and f factors as in (5.5). Because of Corollary 3.12, the cubic factor of f in (5.5) factors as claimed if and only if $M = 0$. □

Let $I = 10a + 2 - 7c + d + e$, $J = a - c$ and $N = 8a^3 + 3ab^2 + 2b^3 + 4a^2d + 2abd + b^2d + ad^2$.

Theorem 5.10. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial as in Lemma 5.1. Then*

$$f = (x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)(A(x+y) + Bz)(A(x+z) + By)(A(z+y) + Bx)$$

where $A, B \in \mathbb{C}$ if and only if $I = J = N = 0$.

Proof. By Theorem 5.7, $(x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)$ is a factor of f if and only if $I = J = 0$ and f factors as in (5.8). Because of Corollary 3.12, the cubic factor of f in (5.8) factors as claimed if and only if $N = 0$. □

Theorem 5.11. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial as in Lemma 5.1. Then*

$$f = (x + \omega y + \omega^2 z)(x + \omega^2 y + \omega z)g$$

for some $g \in \mathbb{C}[x, y, z]$ if and only if $I = J = 0$.

Proof. Suppose $f = (x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)g$. Then g is symmetric by Lemma 1.6 and so $f = (x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)(\alpha(x^3+y^3+z^3)+\beta(xy^2+xz^2+yx^2+yz^2+zx^2+zy^2)+\gamma xyz)$ for some $\alpha, \beta, \gamma \in \mathbb{C}$. Comparing the coefficients of this equation with the form of f in (5.1) we get the following equations.

$$\begin{aligned} a &= \alpha \\ b &= -\alpha+\beta \\ c &= \alpha \\ d &= -\gamma \\ e &= -\alpha-2\beta+\gamma \end{aligned} \tag{5.7}$$

Plugging these equations into I and J we get 0 for both of them.

Now, suppose $I = J = 0$. Solving these equations we get $e = -3a-2b-d$ and $c = a$, and now plugging these equations into f as in (5.1) we get,

$$\begin{aligned} f = &(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z) \\ &(a(x^3+y^3+z^3)+(a+b)(xy^2+xz^2+yx^2+yz^2+zx^2+zy^2)-dxyz). \end{aligned} \tag{5.8}$$

So f factors as claimed. $\square$

Theorem 5.12. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial as in Lemma 5.1.*

(1) $f = a(x+y+z)^3(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$ *if and only if* $b = 2a$, $c = a$, $d = -6a$ *and* $e = -a$.

(2) $f = a(x+y+z)(x+\omega y+\omega^2 z)^2(x+\omega^2 y+\omega z)^2$ *if and only if* $b = -a$, $c = a$, $d = 3a$ *and* $e = -4a$.

Proof. Both of these results are direct consequences of matching coefficients with the equation (5.1). □

Theorem 5.13. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial as in (5.1). Set*

$$\begin{aligned} S &= 5a - b - 2c + e \\ T &= 5a + 3b - 5c + d. \end{aligned} \tag{5.9}$$

Then $f = (x+y+z)gh$, where $\deg g = \deg h = 2$ and g and h are symmetric, if and only if $S = T = 0$.

Proof. Suppose $f = (x+y+z)gh$, where $\deg g = \deg h = 2$ and g and h are symmetric. Then $f = (x+y+z)(s_1(x^2+y^2+z^2)+t_1(xy+xz+yz))(s_2(x^2+y^2+z^2)+t_2(xy+xz+yz))$ for some $s_1, s_2, t_1, t_2 \in \mathbb{C}$. Comparing the coefficients of this equation with the form of f in (5.1) we get the following equations.

$$\begin{aligned} a &= s_1s_2 \\ b &= s_1s_2 + s_2t_1 + s_1t_2 \\ c &= 2s_1s_2 + s_2t_1 + s_1t_2 + t_1t_2 \\ d &= 2s_1s_2 + 2s_2t_1 + 2s_1t_2 + 5t_1t_2 \\ e &= 3s_2t_1 + 3s_1t_2 + 2t_1t_2 \end{aligned} \tag{5.10}$$

Plugging in these equations into S and T we get 0 for both of them.

Now, suppose $S = T = 0$. Solving for e in $S = 0$ and d in $T = 0$ and plugging these equations into f as in (5.1) we get,

$$
\begin{aligned}
f = (x+y+z)(&ax^4 - ax^3y + bx^3y + ax^2y^2 - bx^2y^2 \\
&+ cx^2y^2 - axy^3 + bxy^3 + ay^4 - ax^3z + bx^3z \\
&- 3ax^2yz - bx^2yz + 2cx^2yz - 3axy^2z - bxy^2z \\
&+ 2cxy^2z - ay^3z + by^3z + ax^2z^2 - bx^2z^2 \\
&+ cx^2z^2 - 3axyz^2 - bxyz^2 + 2cxyz^2 + ay^2z^2 \\
&- by^2z^2 + cy^2z^2 - axz^3 + bxz^3 - ayz^3 + byz^3 + az^4) \\
= (x+y+z)\big(&A(x^4+y^4+z^4) + B(x^2y^2 + x^2z^2 + y^2z^2) \\
&+ C(x^3y + x^3z + y^3x + y^3z + z^3x + z^3y) + D(x^2yz + y^2xz + z^2xy)\big),
\end{aligned}
$$

where

$$
\begin{aligned}
A &= a \\
B &= a - b + c \\
C &= -a + b \\
D &= -3a - b + 2c.
\end{aligned}
$$

Because $2B - 4A + C - D = 0$, Theorem 4.9 implies that the degree four factor of f is a product of two degree two symmetric polynomials as claimed. $\square$

Theorem 5.14. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial as in* (5.1). *Set*

$$
\begin{aligned}
R = 25a^3 &+ 5ab^2 + 2b^3 - 40a^2c + 10abc - 2b^2c + 11ac^2 - 10bc^2 \\
&+ 4c^3 + 15a^2d + b^2d - 12acd + 2bcd + ad^2 - 5a^2e - 10abe \\
&- b^2e + 9ace + 7bce - 4c^2e + ade - bde - ae^2 - be^2 + ce^2.
\end{aligned}
$$

Then $f = gh$*, where* $\deg g = 2$*,* $\deg h = 3$ *and* g *and* h *are symmetric, if and only if* $R = 0$*. When this happens,*

$$\begin{aligned} f = \frac{1}{S^2}&(S(x^2+y^2+z^2)+T(xy+xz+yz)) \\ &\cdot \Big(aS(x^3+y^3+z^3)+(bS-aT)(x^2y+xy^2+y^2z+z^2y+xz^2+zx^2) \\ &+((2a+2b-2c+e)S-3aT)xyz\Big) \end{aligned} \tag{5.11}$$

if $S \neq 0$*, or otherwise,*

$$f = (xy+xz+yz)(b(x^3+y^3+z^3)+c(x^2y+xy^2+x^2z+y^2z+xz^2+yz^2)+(2c+d)xyz).$$

Proof. Suppose $f = gh$, where $\deg g = 2$, $\deg h = 3$ and g and h are symmetric. Then, $f = (s(x^2+y^2+z^2)+t(xy+xz+yz))\big(A(x^3+y^3+z^3)+B(xy^2+xz^2+yx^2+yz^2+zx^2+zy^2)+Cxyz\big)$ for some s, t, A, B, $C \in \mathbb{C}$. Comparing the coefficients of this equation with the form of f in (5.1) we get the following equations.

$$\begin{aligned} a &= As \\ b &= Bs + At \\ c &= As + B(s+t) \\ d &= 2B(s+t) + Ct \\ e &= Cs + At + 2Bt. \end{aligned} \tag{5.12}$$

Plugging in these equations into R gives 0.

For the converse, define S and T as in (5.9) and

$$\begin{aligned} F = &(S(x^2+y^2+z^2)+T(xy+xz+yz)) \\ &\cdot \Big(aS(x^3+y^3+z^3)+(bS-aT)(x^2y+xy^2+y^2z+z^2y+xz^2+zx^2) \\ &+((2a+2b-2c+e)S-3aT)xyz\Big). \end{aligned}$$

A tedious calculation shows that $S^2 f = F + R(x+y+z)(xy+yz+zx)^2$. If $R = 0$ and $S \neq 0$, then it is clear that f is a scalar multiple of F. Hence, f factors as claimed. We are now left with the case that $S = R = 0$. Solving for e in $S = 0$ and plugging this condition into R we get that $R = aT^2$. Therefore $a = 0$ or $T^2 = 0$. If $T = 0$, Theorem 5.13 implies that f factors as expected. If $a = 0$ then $f = (xy+xz+yz)(b(x^3+y^3+z^3)+c(x^2y+xy^2+x^2z+y^2z+xz^2+yz^2)+(2c+d)xyz)$ and f factors as expected. $\square$

This theorem tells you when f factors as a degree 2 factor and a degree 3 factor. To determine when the degree 3 factor is completely reducible, we define

$$\begin{aligned} P = {} & 8b^3 + 50a^2c - 70abc + 20b^2c + 20ac^2 - 14bc^2 + 2c^3 + 10abd + 2b^2d - 5acd \\ & + bcd - c^2d + 5ad^2 - bd^2 - 25a^2e + 20abe - 19b^2e + 6bce + c^2e \\ & - 15ade + bde + cde + 10ae^2 + 2be^2 - 2ce^2 \end{aligned}$$

$$\begin{aligned} Q = {} & 25a^4 + 10a^3b - 5a^2b^2 + 2ab^3 + 35a^3c - 28a^2bc + 13ab^2c - 2b^3c - 29a^2c^2 \\ & + 18abc^2 - 4b^2c^2 - 4ac^3 - 5a^3d + 3a^2bd + 9a^2cd - 2abcd - 15a^3e + 9a^2be \\ & - 7ab^2e + b^3e + 7a^2ce - 13abce + 4b^2ce + 8ac^2e - 5a^2de + abde + 5a^2e^2 \\ & + 2abe^2 - b^2e^2 - 5ace^2 + ae^3. \end{aligned}$$

Theorem 5.15. *Let f be a trivariate, homogeneous, degree 5 and symmetric polynomial as in (5.1). Then*

$$f = (\alpha x + \beta(y+z))(\alpha y + \beta(x+z))(\alpha z + \beta(y+x))g \tag{5.13}$$

for some $\alpha, \beta \in \mathbb{C}$ and $g \in \mathbb{C}[x,y,z]$ if and only if $R = P = Q = 0$.

Proof. Suppose $f = (\alpha x + \beta(y+z))(\alpha y + \beta(x+z))(\alpha z + \beta(y+x))g$, where g is a symmetric degree two polynomial of the form $a_2(x^2+y^2+z^2) + b_2(xy+yz+xz)$, for some $a_2, b_2 \in \mathbb{C}$. Equating coefficients of f as given by (5.1) and (5.13) we get

$$
\begin{aligned}
a &= \alpha a_2 \beta^2 \\
b &= \alpha^2 a_2 \beta + \alpha a_2 \beta^2 + a_2 \beta^3 + \alpha \beta^2 b_2 \\
c &= \alpha^2 a_2 \beta + 2\alpha a_2 \beta^2 + a_2 \beta^3 + \alpha^2 \beta b_2 + \alpha \beta^2 b_2 + \beta^3 b_2 \\
d &= 2\alpha^2 a_2 \beta + 2\alpha a_2 \beta^2 + 2a_2 \beta^3 + \alpha^3 b_2 + 2\alpha^2 \beta b_2 + 5\alpha \beta^2 b_2 + 4\beta^3 b_2 \\
e &= \alpha^3 a_2 + 3\alpha a_2 \beta^2 + 2a_2 \beta^3 + 2\alpha^2 \beta b_2 + 3\alpha \beta^2 b_2 + 2\beta^3 b_2.
\end{aligned}
$$

Plugging these equations into R, P, and Q gives 0 in each case.

Conversely, suppose $P = Q = R = 0$. Let

$$F = (Ax+B(y+z))(Ay+B(x+z))(Az+B(y+x))(S(x^2+y^2+z^2)+T(xy+xz+yz))$$

where $A = 5a^2 - 5ab - 9ac + 2bc + 4c^2 - ad + 7ae - be - 4ce + e^2$ and $B = -20a^2 + 5ab - b^2 + 11ac - 2bc - ad - 3ae + be$, with S and T as defined in (5.9). A tedious calculation without the assumption that $P = Q = R = 0$ gives

$$S^6 f = F + R(a^2 T + 2a(4a-b)S)H_1 + S^4 R H_2 + S^2(Q - aR)H_1$$

where

$$
\begin{aligned}
H_1 &= (S(x^2+y^2+z^2) + T(xy+xz+yz))(x+y+z)^3 \\
H_2 &= (x+y+z)(xy+xz+yz)^2.
\end{aligned}
$$

If $S \neq 0$ and $Q = R = 0$ then this equation implies $f = F/S^6$ and f has the claimed the form. (Notice that if $S \neq 0$ then the condition $P = 0$ is not needed.)

Suppose now that $S = 0$. As in the proof of Theorem 5.14, the condition $R = S = 0$ implies that $a = 0$ or $T = 0$.

(1) Suppose $S = T = P = 0$. Solving the equations $S = T = 0$ for d and e we get $e = -5a + b + 2c$ and $d = -5a - 3b + 5c$. Plugging these equations into $P = 0$ and factoring gives $(-5a+3b-c)^3 = 0$, which can be rewritten as $c = -5a+3b$. Plugging in these expressions for c, d and e into f as in (5.1) we get

$$f = (x + y + z)^3(a(x^2 + y^2 + z^2) + (-3a + b)(xy + xz + yz))$$

and so, f has the claimed form of (5.13) with $\alpha = \beta$.

(2) Suppose $S = a = P = 0$. Solving $S = 0$ for e we get $e = -5a+b+2c$. Plugging these equations into f we get

$$f = (xy+xz+yz)(b(x^3+y^3+z^3)+c(x^2y+xy^2+y^2z+z^2y+xz^2+zx^2)+(d-2c)xyz).$$

Now by Corollary 3.12, the degree 3 factor of f here has the form $(\alpha x + \beta(y + z))(\alpha y + \beta(x + z))(\alpha z + \beta(y + x))$ for some $\alpha, \beta \in \mathbb{C}$ if and only if $9b^3 + 6b^2c + bc^2 + 4c^3 - 3b^2d - 4bcd - c^2d + bd^2 = 0$. Setting $a = 0$ and $c = -5a + 3b$ in $P = 0$ gives us this exact same equation. $\square$

In Table 5.1 we have collected all of the results in Theorems 5.7–5.15.

Theorem 5.16. *Let*

$$\begin{aligned} f = a(x^5 + y^5 + z^5) &+ b(x^4y + x^4z + y^4x + y^4z + z^4x + z^4y) \\ &+ c(x^3y^2 + x^3z^2 + y^3x^2 + y^3z^2 + z^3x^2 + z^3y^2) \\ &+ d(x^2y^2z + z^2y^2x + z^2x^2y) + e(x^3yz + y^3xz + z^3xy) \end{aligned}$$

where $a, b, c, d, e \in \mathbb{C}$. *Set*

$$W = 5a - 5b + c - d + 2e \qquad U = 5a - 3b + c \qquad V = 15a - 7b + e$$

$$I = 10a + 2b - 7c + d + e \qquad J = a - c \qquad K = -2a + b - e - c$$

$$S = 5a - b - 2c + e \qquad T = 5a + 3b - 5c + d$$

$$L = (a-b)^2 + ae$$

$$M = 25a^3 - 40a^2b + 21ab^2 - 6b^3 - 5a^2e + 4abe + b^2e - ae^2$$

$$N = 8a^3 + 3ab^2 + 2b^3 + 4a^2d + 2abd + b^2d + ad^2$$

$$\begin{aligned} R = {} & 25a^3 + 5ab^2 + 2b^3 - 40a^2c + 10abc - 2b^2c + 11ac^2 - 10bc^2 + 4c^3 + 15a^2d \\ & + b^2d - 12acd + 2bcd + ad^2 - 5a^2e - 10abe - b^2e + 9ace + 7bce - 4c^2e \\ & + ade - bde - ae^2 - be^2 + ce^2 \end{aligned}$$

$$\begin{aligned} P = {} & 8b^3 + 50a^2c - 70abc + 20b^2c + 20ac^2 - 14bc^2 + 2c^3 + 10abd + 2b^2d - 5acd \\ & + bcd - c^2d + 5ad^2 - bd^2 - 25a^2e + 20abe - 19b^2e + 6bce + c^2e \\ & - 15ade + bde + cde + 10ae^2 + 2be^2 - 2ce^2 \end{aligned}$$

$$\begin{aligned} Q = {} & 25a^4 + 10a^3b - 5a^2b^2 + 2ab^3 + 35a^3c - 28a^2bc + 13ab^2c - 2b^3c - 29a^2c^2 \\ & + 18abc^2 - 4b^2c^2 - 4ac^3 - 5a^3d + 3a^2bd + 9a^2cd - 2abcd - 15a^3e + 9a^2be \\ & - 7ab^2e + b^3e + 7a^2ce - 13abce + 4b^2ce + 8ac^2e - 5a^2de + abde + 5a^2e^2 \\ & + 2abe^2 - b^2e^2 - 5ace^2 + ae^3 \end{aligned}$$

Then f *is reducible if and only if* $W = 0$ *or* $R = 0$. *The possible factorizations of* f *and when each occurs are in Table 5.1.*

	Factorization	Equivalent Condition
1	$f = (x+y+z)\,g$	$W = 0$
2	$f = (x+y+z)^2\,g$	$W = U = 0$
3	$f = (x+y+z)^3\,g$	$W = U = V = 0$
4	$f = a(x+y+z)^5$	$a = \dfrac{b}{5} = \dfrac{c}{10} = \dfrac{d}{30} = \dfrac{e}{20}$
5	$f = (x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)h$	$I = J = 0$
6	$f = a(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)(x+y+z)^3$	$a = \dfrac{b}{2} = c = -\dfrac{d}{6} = -e$
7	$f = a(x+\omega y+\omega^2 z)^2(x+\omega^2 y+\omega z)^2(x+y+z)$	$a = -b = c = \dfrac{d}{3} = -\dfrac{e}{4}$
8	$f = (\alpha(\omega x^2+y^2+\omega^2 z^2)+\beta(\omega^2 xy+\omega yz+xz))$ $\times\,(\alpha(\omega^2 x^2+y^2+\omega z^2)+\beta(\omega xy+\omega^2 yz+xz))$ $\times\,(x+y+z)$	$W = L = K = 0$
9	$f = (\alpha x+\beta(y+z))(\alpha y+\beta(x+z))(\alpha z+\beta(y+x))$ $\times\,(x+y+z)^2$	$W = U = M = 0$
10	$f = (\alpha x+\beta(y+z))(\alpha y+\beta(x+z))(\alpha z+\beta(y+x))$ $\times\,(x+\omega y+\omega^2 z)(x+\omega^2 y+\omega z)$	$I = J = N = 0$
11	$f = (x+y+z)\,g\,h$ with $\deg g = \deg h = 2$	$S = T = 0$
12	$f = gh$ with $\deg g = 3$ and $\deg h = 2$	$R = 0$
13	$f = (\alpha x+\beta(y+z))(\alpha y+\beta(x+z))(\alpha z+\beta(y+x))g$	$R = P = Q = 0$

Table 5.1: Factorization of degree five polynomials

Proof. Since $\deg f = 5$ and f is reducible, f must factor into one degree one polynomial and a degree four polynomial, five degree one polynomials, three degree one polynomials and one degree two polynomials, two degree two polynomials and a degree one polynomial, one degree three polynomial and a degree two polynomial, or one degree three polynomial and two degree one polynomials (assuming each polynomial in the factorization of f is irreducible).

(1) Suppose f is a product of a degree one irreducible polynomial and a degree four irreducible polynomial. By Theorem 5.5(1), the degree one polynomial is symmetric, and is a scalar multiple of $x + y + z$ by Lemma 1.14(2). Now, by Theorem 5.7, $W = 0$ and hence f factors as in line 1 of the table.

(2) Suppose f is a product of five degree one polynomials. Now, by Theorem 5.2, f factors in one of three ways: If f factors as Theorem 5.2(A), then f factors as in line 9 of the table. If f factors as Theorem 5.2(B), then f factors as in line 10 of the table. If f factors as Theorem 5.2(C), then f factors as in either line 4, line 6 or line 7 of the table.

(3) Suppose f is a product of three degree one polynomials and one degree two polynomials. Let $f = gh$ where g is the product of the degree one polynomials. According to Theorem 5.6, g factors as described by Theorem 3.8. If g factors as Theorem 3.8(1), then f factors as in line 3 of the table. If f factors as Theorem 3.8(2), then f factors as in line 5 of the table. If f factors as Theorem 3.8(3), then f factors as in line 13 of the table.

(4) Suppose $f = ghk$ where g, h and k are irreducible and $\deg g = \deg h = 2$

and $\deg k = 1$. Then by Theorem 5.3, f is a product of two degree two irreducible polynomials and a degree one irreducible polynomial and k is symmetric. By Theorem 1.12, k is symmetric, and is a scalar multiple of $x+y+z$ by Lemma 1.14(2). Therefore f factors as in line 11 of the table with $S=T=0$.

(5) Suppose f is a product of one degree three polynomial and a degree two polynomial. According to Theorem 5.14, this occurs when $R=0$ and f factors as in line 12 of the table.

(6) Suppose f is a product of one degree three polynomial and two degree one polynomials. By Theorem 5.5(2), the product of the two degree one polynomials is symmetric and hence, by Theorem 2.5, is a scalar multiple of $(x+\lambda y+\lambda^2 z)(x+\lambda^2 y+\lambda z)$ where $\lambda^3=1$. If $\lambda=1$, f factors as in line 2 of the table, and if $\lambda \neq 1$, then f factors as line 5 of the table.

It remains to show that in all of these factorizations, either $W=0$ or $R=0$. If f factors as in lines 1, 2, 3, 4, 6, 7, 8, 9 or 11 of the table, then $x+y+z$ divides f and so $W=0$. If f factors as in line 12 or line 13 of the table, then $R=0$. If f factors as in line 5 or line 10 of the table, then f is a product of a degree two and a degree three polynomial, so $R=0$. $\square$

Example 5.17. *Consider the polynomial* $f = x^5+y^5+z^5+x^3y^2+x^2y^3+x^3z^2+y^3z^2+x^2z^3+y^2z^3$. *For this polynomial* $a=c=1$ *and* $b=d=e=0$. *Plugging these values into the formula for* R *we find that* $R=0$. *As a result, from line* 12 *of the table, we know that* f *is a product of a degree* 3 *and degree* 2 *polynomial. Does* f *factor further?*

No. Plugging in the values we find that $W = 15 \neq 0$, *and so* $x + y + z$ *does not divide* f *and consequently* f *does not factor as in lines* 1, 2, 3, 4, 6, 7, 8, 9 *or* 11 *of the table. Since* $I = 3 \neq 0$, f *does not factor as lines* 5 *and* 10 *of the table. Since* $Q = 27 \neq 0$, f *does not factor as line* 13 *of the table. Since all possible factorizations of* f *are in the table, we now know that* f *does not factor further.*

Since $R = 0$, *we can use Theorem* 5.14 *to find the factors of* f. *Because* $S = 3$ *and* $T = 0$, (5.11) *implies that*

$$f = (x^2 + y^2 + z^2)(x^3 + y^3 + z^3).$$

As expected, these factors of f *are irreducible.*

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