TO WHAT EXTENT IS THE KAZHDAN CONSTANT A GRAPH INVARIANT?

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ABSTRACT

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In this thesis, we investigate the Kazhdan constant of groups and generating sets associated with Cayley graphs. In particular, we look at groups and generating sets that yield isomorphic Cayley graphs and compute their Kazhdan constants–in other words we are asking if the Kazhdan constant is a Cayley graph invariant. We give explicit formulas for the Kazhdan constant of the group $\mathbb{Z}_n$ with generating set $\{1, n-1\}$ and the dihedral group D_n, which has order $2n$, with generating set $\{s, sr\}$, where s is a reflection and r is a rotation. We also find a sufficient condition for the Kazhdan constant to be a Cayley graph invariant.

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TABLE OF CONTENTS

LIST OF FIGURES

Figure

CHAPTER 1

Introduction

1.1 Overview

In Chapter 1 of this thesis we give an overview of the thesis and a brief discussion about the Kazhdan constant. Chapter 2 will encompass the basic background we need from graph theory in order to make our problem more tangible for the reader. We discuss properties of Cayley graphs, the adjacency matrix associated to a graph, the isoperimetric constant of a graph, and most importantly expander families. Chapter 3 is the study of representation theory and character theory. We again go over basic results in this field with examples that will later aid us in solving our main question at hand. Chapter 4 is the study of the Kazhdan constant and its properties. We derive useful properties and state known results which relate the Kazhdan constant to other graph theory invariants. Our last chapter will focus on the main result and some conjectures we propose for the next generation of scholars.

Remark 1.1. We expect the reader to have a sufficient knowledge of group theory and basic knowledge in graph theory. Readers inclined to learn the needed graph theory can refer to [7]. We include the theory of graphs in order to obtain a tangible application of our main result to the reader.

1.2 Significance

Expander families (certain sequences of graphs) are a subject of much interest, both in their own right, and also because of their many applications in computer science. See, for example, [6] for a survey of this topic. One way to produce expander families is through the Cayley graph construction, in which a group G and a symmetric subset Γ of G are used to create a graph. Every such pair (G, Γ) has an associated *Kazhdan constant* $\kappa(G, \Gamma)$, as defined in Section 4.1 below. As discussed in [6], one common method for studying the expansion properties of a sequence of Cayley graphs is to look at the Kazhdan constants of the pairs (G, Γ) used to create the graphs. Hence our interest in Kazhdan constants.

It is possible for two different pairs (G_1, Γ_1) and (G_2, Γ_2) to produce isomorphic Cayley graphs. The book [7] raises the question: To what extent is the Kazhdan constant a graph invariant? In particular, if (G_1, Γ_1) and (G_2, Γ_2) produce isomorphic Cayley graphs, must we then have $\kappa(G_1, \Gamma_1) = \kappa(G_2, \Gamma_2)$? In this thesis, we answer that question in the positive way, by exhibiting infinitely many examples of pairs (G_1, Γ_1) and (G_2, Γ_2) which yield isomorphic Cayley graphs and $\kappa(G_1, \Gamma_1) = \kappa(G_2, \Gamma_2)$. The groups in question are cyclic and dihedral groups, and we choose Γ_1 and Γ_2 such that the corresponding Cayley graphs are cycle graphs.

Kazhdan constants are defined in terms of representations of the underlying group. We assume familiarity with basic representation theory.

CHAPTER 2

Graph Theory

The study of graph theory is a deep and rich subject, from the birth of graph theory due to Euler up until the recent discovery of showing the chromatic number of the plane is at least five. This thesis will only discuss a small sample of topics in the field of graph theory which are pertinent to our main discussion. This chapter builds up to the interplay between representation theory and graph theory, so we want the reader to have exposure to how the Kazhdan constant plays a role in graph theory.

This chapter is structured as follows. We define what a Cayley graph is and give examples and properties. We then look at the associated matrix which stems from a graph. In the same section we discuss a conjecture stated by Ádám [1] that gives another form of isomorphism related to Cayley graphs. The last section discusses expander families, which relates to the core of the thesis.

2.1 What are Cayley Graphs?

The definition of Cayley graphs gives an algorithm to construct a graph from a group and a subset of that group. Besides this useful algorithm to generate a graph, Cayley graphs are used because we can use properties in group theory to tell us information about the graph. We give the following definition that describes this type of graph below.

Remark 2.1. Note that we will only consider simple graphs in this thesis.

Definition 2.2 (vertex set, edge set)**.** *A graph is composed of a* ***vertex set*** V_x *and an*

edge set E. *Let* $v, w \in V_x$. *If* $\{v, w\} \in E$, *then we say that* v *and* w *are* ***adjacent***.

Definition 2.3 (graph isomorphism)**.** *An* ***isomorphism*** *from a graph* G *to a graph* H *is a bijection* $f : V(G) \to V(H)$ *such that* $uv \in E(G)$ *if and only if* $f(u)f(v) \in E(H)$. *We say "*G *is* ***isomorphic to*** H*" if there is an isomorphism from* G *to* H.

Definition 2.4 (symmetric subset)**.** *Let* G *be a group and* Γ *a subset of* G. *We say that* Γ *is a* ***symmetric*** *if whenever* γ *is an element of* Γ, *then* γ^{-1} *is an element of* Γ. *We denote a symmetric subset by the symbol* $\subseteq_s$.

Definition 2.5 (Cayley Graph)**.** *Let* G *be a group and* $\Gamma \subseteq_s G$. *The* ***Cayley graph*** *of* G *with respect to* Γ, *denoted by* $Cay(G, \Gamma)$, *is defined as follows. The vertices of* $Cay(G, \Gamma)$ *are the elements of* G. *Two vertices* $x, y \in G$ *are adjacent if and only if there exists* $\gamma \in \Gamma$ *such that* $x = \gamma y$. *In other words,* $y^{-1}x \in \Gamma$.

We give a couple of examples to illustrate the definition of a Cayley graph.

Example 2.6. We compute the Cayley graph of the group $\mathbb{Z}_6$ with subset $\{1, 5\}$.

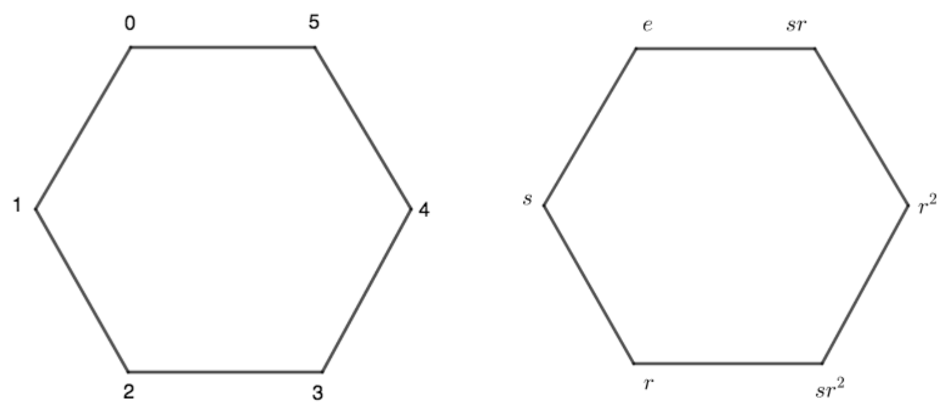
adj. vert.	1	$x \sim y$	5	$x \sim y$
0	$0+1=1$	$0 \sim 1$	$0+5=5$	$0 \sim 5$
1	$1+1=2$	$1 \sim 2$	$1+5=0$	$1 \sim 0$
2	$2+1=3$	$2 \sim 3$	$2+5=1$	$2 \sim 1$
3	$3+1=4$	$3 \sim 4$	$3+5=2$	$3 \sim 2$
4	$4+1=5$	$4 \sim 5$	$4+5=3$	$4 \sim 3$
5	$5+1=0$	$5 \sim 0$	$5+5=4$	$5 \sim 4$

Refer to Figure 2.1(a).

Example 2.7. We compute the Cayley graph of the group D_3 with subset $\{s, sr\}$ where s denotes the reflection about the x-axis and r a rotation of 120 degrees.

adj. vert.	s	$x \sim y$	sr	$x \sim y$
e	$e(s) = s$	$e \sim s$	$e(sr) = sr$	$e \sim sr$
r	$r(s) = sr^2$	$r \sim sr^2$	$r(sr) = s$	$r \sim s$
r^2	$r^2(s) = sr$	$r^2 \sim sr$	$r^2(sr) = sr^2$	$r^2 \sim sr^2$
s	$s(s) = e$	$s \sim e$	$s(sr) = r$	$s \sim r$
sr	$sr(s) = r^2$	$sr \sim r^2$	$sr(sr) = e$	$sr \sim e$
sr^2	$sr^2(s) = r$	$sr^2 \sim s$	$sr^2(sr) = r^2$	$sr^2 \sim r^2$

Refer to Figure 2.1(b).



(a) $Cay(\mathbb{Z}_6, \{1, 5\})$ (b) $Cay(D_3, \{s, sr\})$

Figure 2.1: Comparing Cayley graphs from our previous two examples.

Remark 2.8. From the previous two examples we can notice some trends about Cayley graphs. Note that even though our corresponding groups are not isomorphic, the Cayley graphs of non-isomorphic groups can be isomorphic Cayley graphs. Also, does it always have to be the case that the if the subset of the group generates the group then the Cayley graph associated to those pairs forms a connected graph? A last note, is every Cayley graph $|\Gamma|$-regular where Γ is the generating set? Are these conjectures true for all Cayley graphs or are these cycle graphs a special case of this phenomenon? It turns out these facts are true for any Cayley graphs. We present the propositions below.

Proposition 2.9. *Let G be a finite group and Γ a symmetric subset of the group with cardinality d. Then the following properties hold:*

(1) *$Cay(G, \Gamma)$ is $|d|$-regular.*

(2) *$Cay(G, \Gamma)$ is connected if and only if Γ generates G.*

Remark 2.10. We note that not every regular graph is a Cayley graph. The canonical example is the Peterson graph.

Proposition 2.11. *Let G be a finite group and $\Gamma \Subset G$. Suppose $Cay(G, \Gamma)$ is a cycle graph then G is a either a cyclic group or dihedral group.*

The proofs of proposition 2.9 and 2.11 are straightforward and left to the reader.

2.2 Ádám's Conjecture

In 1967, A. Ádám conjectured necessary and sufficient conditions for two Cayley graphs of a cyclic group of order n to be isomorphic. In 1969, the first counterexample was given by Bernard Elspas and James Turner (1969)[4]. In the same paper they give conditions for n that make Ádám's conjecture hold. Pàlfy[9] later gives necessary and sufficient conditions for when Ádám's conjecture holds. In section 5.2 we create a family of Cayley graphs for which Ádám's conjecture fails.

This section is devoted to the statement of Ádáms conjecture. To state it, we must define "Cayley isomorphism" which plays an important role in later chapters. We first state Ádám's conjecture in its original form. We then rephrase it using more general group–theoretical language.

Definition 2.12 (Adjacency Matrix)**.** *Let X be a graph with an ordering of its vertices given by $v_1, v_2, \ldots, v_n$. Then the* ***adjacency matrix*** *for X is the matrix A, where $A_{i,j}$ is the number of edges that are incident to both v_i and v_j.*

Example 2.13. We find the adjacency matrix of $Cay(\mathbb{Z}_6, \{1, 5\})$. Looking back at Figure 2.1 we can determine the adjacency matrix of our Cayley graph, which is:

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 & 1 \\ 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}$$

with respect to the natural ordering of the vertices.

Remark 2.14. We notice from the previous example that the rows of the matrix are cyclic permutations of each other. This motivates a definition.

Definition 2.15. *A matrix C is called* ***circulant*** *if it can be written in the form*

$$C = \begin{pmatrix} c_0 & c_1 & c_2 & \cdots & c_{n-1} \\ c_{n-1} & c_0 & c_1 & \cdots & c_{n-2} \\ c_{n-2} & c_{n-1} & c_0 & \cdots & c_{n-3} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ c_1 & c_2 & c_3 & \cdots & c_0 \end{pmatrix}$$

Remark 2.16. We note two things. Since we can create an adjacency matrix associated to a graph we can talk about eigenvalues. Also, since we are only working with Cayley graphs, which are simple, with symmetric subset our adjacency matrix will be symmetric, hence we get that the eigenvalues are all real. Now for our research, it turns out that for regular graphs the most interesting and useful eigenvalue is the second-largest eigenvalue. It turns out that the second-largest eigenvalue is strongly related to the Kazhdan constant. Readers can refer to [7] if they are inclined to learn more about how the second largest eigenvalue interacts with the Kazhdan constant.

Definition 2.17. *Let G be a graph with n vertices, and let $0 < \gamma_1 < \ldots < \gamma_m < n$ be a given set of m integers. The set $\Gamma = \{\gamma_1, \ldots, \gamma_m\}$ is said to be called the* ***connection set*** *of G if the adjacency matrix of G is a circulant matrix with 1's in positions $\gamma_1, \ldots, \gamma_m$ of the first row.*

Definition 2.18. *Let G be a graph with n vertices, and let $\Gamma_1 = \{\gamma_1, \ldots, \gamma_m\}$ be a connection set of G. If there exists an integer r relatively prime to n such that $\Gamma_2 = r\Gamma_1 = \{r\gamma_i : \gamma_i \in \Gamma_1\}$, then this is called* ***equivalence of connection set****.*

Conjecture 2.19. Two graphs with circulant adjacency matrices are isomorphic if and only if there is an equivalence of their connection sets.

With a little deciphering of the language used in the conjecture we get a modified version of Adams conjecture. We state a necessary definition to help make this modified conjecture come to life. It turns out this idea will also play a pivotal role in the upcoming chapters.

There is a straightforward sufficient condition for when Cayley graphs are isomorphic. The proof is left to the reader.

Proposition 2.20. *Let G be a group. Let $\Gamma_1, \Gamma_2 \subseteq G$. If there exists $\phi \in Aut(G)$ such that $\phi(\Gamma_1) = \Gamma_2$, then $Cay(G, \Gamma_1) \cong Cay(G, \Gamma_2)$.*

Definition 2.21. *Let G be a group. Let $\Gamma_1, \Gamma_2 \subset G$. We say (G, Γ_1) and $(G, \Gamma_2)$are* ***Cayley isomorphic*** *if there exists an automorphism ϕ of G such that $\phi(\Gamma_1) = \Gamma_2$.*

Definition 2.22. *A group G is called a* ***CI-group*** *if, for any subsets $\Gamma_1, \Gamma_2 \subset G$, whenever $Cay(G, \Gamma_1) \cong Cay(G, \Gamma_2)$, then there exists an $\phi \in Aut(G)$ such that $\phi(\Gamma_1) = \Gamma_2$.*

Remark 2.23. Àdàm's conjecture precisely states that every finite cyclic group is a CI-group. As it turns out, there are necessary and sufficient conditions for Àdàm's conjecture to be true.

Theorem 2.24. *$\mathbb{Z}_n$ is a CI-group if and only if $(n, \varphi(n)) = 1$ or $n = 4$, where φ in this case is the Euler function [9].*

2.3 Introduction to Expander Families

We think of expander families as being a "good" communication network, i.e., reliable, fast while still being cost-effective. There is a strong connection regarding the isoperimetric constant and the second- largest eigenvalue. Readers can refer to [7] if interested. The isoperimetric constant plays a pivotal role in making the Kazhdan constant palpable to graph theorist.

Definition 2.25 (Boundary)**.** *Let X be a graph with vertex set V. Let $F \subset V$. The **boundary** of F, denoted by ∂F, is defined to be the set of edges with one endpoint in F and one endpoint in F^c. That is, ∂F is the set of edges connecting to F to F^c.*

Definition 2.26 (Isoperimetric constant)**.** *The **isoperimetric constant** of a graph X with vertex set V is defined as*

$$h(X) = \min \left\{ \frac{|\partial F|}{|F|} : F \subset V \text{ and } |F| \leq \frac{|V|}{2} \right\}.$$

Definition 2.27 (Expander Family)**.** *Let d be a positive integer. Let (X_n) be a sequence of d-regular graphs such that $|X_n| \to \infty$ as $n \to \infty$. We say that (X_n) is an **expander family** if the sequence $(h(X_n))$ is bounded away from zero.*

Let us consider a cycle graph of order n when n is even. If $|F| = \frac{n}{2}$ and the vertices of F form a path in the cycle graph, then $|\partial F| = 2$. Hence, $h(C_n) \leq \frac{4}{n}$.

Thus, $\lim_{n\to\infty} h(C_n) = 0$. Therefore by Definition 2.39, cycle graphs of even order do not form expander families. A similar argument works for odd cycle graphs also.

Remark 2.28. We notice that cycle graphs do not form an expander family. In other words, they are bad communication networks— which was already apparent in the structure of the graphs. They are neither fast, nor reliable as communication networks.

CHAPTER 3

Representation Theory

In the world of mathematics it is extremely difficult to pinpoint the manifestation of an idea, but in the case of representation theory historians were able to accomplish this task. The birth of representation theory came from a correspondence from Frobenius to Dedekind via letters starting April 12, 1896. The content of these letters discussed the factoring of "group determinants". It was in this exchange that character theory came forth and helped solidify the idea of a representation. Now the general style of teaching representation theory is to define a representation and then from this create the definition of a character, while Frobenius' original approach was the opposite. [8]

In this chapter we work with the ideas of representations of a finite group and the characters of those representations. The first section gives the basic definition of a representation with some examples. We also state a theorem by Maschke that is a well-known structure theorem in algebra. The second section is devoted to the characters of these representations and how they play a pivotal role in constructing irreducible representations. Lastly, we give classical examples of the irreducible representations of the dihedral and cyclic group; we also use character theory to aid us in finding these irreducible representations. We warn the reader that this material is not self-contained. These are merely tools that we need to help us solve our main problem. Many of the pivotal proofs of the theory will not be presented. We incorporated what we deemed necessary for each subject in order for the reader to familiarize themselves with the techniques. Another reason why we choose these results in particular is because we find them beautiful results in the world of mathematics. Readers inclined to see a proof of these results can refer to [7] or [10].

3.1 Irreducible Representations of a Finite Group

This section will include examples and definitions that are pertinent to the discussion of the Kazhdan constant. A global restriction we impose in this paper is that the vector spaces are finite-dimensional; the field we are considering is the complex numbers; and lastly that our group is finite.

Definition 3.1. *Let G be a finite group. A **linear representation** of G is a group homomorphism $\rho : G \to GL(V)$, where V is a finite-dimensional vector space over $\mathbb{C}$. We define the **degree** of ρ or the **dimension** of ρ to be the dimension of V as a vector space over $\mathbb{C}$.*

Definition 3.2. *Let $n \in \mathbb{N}$. A **matrix representation** of G is a group homomorphism from G to $GL(n, \mathbb{C})$.*

Definition 3.3. *We say that $\rho : G \to GL(V)$ and $\rho' : G \to GL(V')$ are **equivalent (or similar) representations** if there is a linear isomorphism $\phi : V \to V'$ such that*

$$\rho'(g) \circ \phi = \phi \circ \rho(g).$$

Remark 3.4. From the definition of equivalent representations, we notice that linear representations and matrix representations coincide for a finite-dimensional vector space. This is evident by fixing a basis of V to obtain an isomorphism from $GL(V)$ to $GL(n, \mathbb{C})$. Most of the computations will be done considering matrix representations, while most theorems will be stated in the context of a linear representation.

Example 3.5. We find two representations for the Klein group.

We will construct two representations. Let $K4G$ be the Klein group: $K4G = \{e, a, b, ab\}$. Let us look at the case when our general linear group is the group of units of the complex numbers, i.e $GL_1(\mathbb{C}) = \mathbb{C}^*$. So our goal is to find two homomorphisms $\rho_1, \rho_2 : G \to \mathbb{C}^*$. If our map, ρ, is a homomorphism then it must be the case that $\rho(e) = 1$. Now if $\rho(a) \in \mathbb{C}^*$ then $|\rho(a)| = 1$ or 2 and so $\rho(a) = 1$ or $\rho(a) = -1$. Let us consider the case when $|\rho(a)| = 1$. Now $\rho(b) \in \mathbb{C}^*$, so $|\rho(b)| = 1$ or 2.

Subcase(1): $\rho(b) = 1$, so $\rho(ab) = \rho(a)\rho(b)$ but then $\rho(ab) = 1$.

Subcase(2): $\rho(b) = -1$, so $\rho(ab) = \rho(a)\rho(b) = 1\rho(b) = \rho(b) = -1$.

Now the possible functions: $\rho_1(x) = 1$ for all $x \in K4G$ and $\rho_2(e) = \rho_2(a) = 1$ and $\rho_2(b) = \rho_2(ab) = -1$ could be homomorphisms. The reader is left to show that in fact they are. Thus, these two functions are representations from the Klein four group to $GL_1(\mathbb{C}^*)$. Note that $\mathrm{Im}(\rho_2) = \langle -1 \rangle$ which is a subgroup of the group of units of the complex numbers.

Remark 3.6. First things first. We call ρ_1 the trivial representation, i.e. the trivial homomorphism, which is always a representation for each group. Now the technique described above is important because it is one way to construct a representation. It is not the most efficient way but at least we have a possible method. This technique also shows why we work with finite groups.

Example 3.7. Is it possible to find a non-trivial n–dimensional representation of the Klein group, where n is an integer greater then one?

The question is asking if there exists a function ρ such that $\rho : G \to GL_n(\mathbb{C})$

is a homomorphism. Let us look at the case when $n = 2$. Following the same logic as above we are left to find which elements have order 2 in $GL_2(\mathbb{C})$. Below we have one:

$$\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

Note that it doesn't matter which element of order two we choose, so long as we stay consistent through out the construction of the representation. Therefore, following the same logic as the previous example, one representation is defined as below:

$$\rho(e) = \rho(a) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \rho(b) = \rho(ab) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

The reader can now generalize this idea to n dimensions to construct representations.

Example 3.8. We will find a non-trivial representation for the cyclic group of order four.

Let us not work too hard on this one. Intuitively, the elements of $\mathbb{Z}_4$ are the rotations of a square that leave it invariant. Hence, let $\rho(a) = e^{\frac{a\pi i}{2}}$. This is in fact a homomorphism from $\mathbb{Z}_4 \to \mathbb{C}^*$. Let $a, b \in \mathbb{Z}_4$ then $\rho(a+b) = e^{\frac{(a+b)\pi i}{2}} = e^{\frac{a\pi i}{2}} \cdot e^{\frac{b\pi i}{2}} = \rho(a) \cdot \rho(b)$ which shows that ρ is a representation.

Remark 3.9. The reader might be wondering at this point if there are an infinite number of non-trivial representations for a finite group. If we look at special representations, called irreducible representations, we can answer this question in a surprising way. The answer to this question will be in the next section.

Definition 3.10. *Let G be a finite group and $\rho : G \to GL(V)$ be a representation of G. We say that a subspace W of V is a* ***G-invariant subspace*** *if $\rho(g)w \in W$ for* *all $g \in G$ and $w \in W$. The G-invariant subspaces $\{0\}$ and V are called the* ***trivial invariant subspace*** *of V. We say that V is* ***reducible*** *if it contains a nontrivial G-invariant subspace. Otherwise, we say V is* ***irreducible****.*

Remark 3.11. By the definition of an irreducible representation we automatically get that *every one-dimensional representation is an irreducible representation.*

Example 3.12. Which representations, from the previous examples, are irreducible representations? We know all one-dimensional representations are irreducible representations. We look at example 3.6. Every representation is reducible in that example when $n \geq 2$. Consider the vector subspace $Span\{(1, 1, \ldots, 1)^t\}$ with where $(1, 1, \ldots, 1)^t\} \in \mathbb{C}^n$.

Definition 3.13. *Let $\rho : G \to GL(V)$ be a representation of a group G. If the vector space under consideration, V, has an inner product such that:*

$$\langle \rho(g)v, \rho(g)w \rangle = \langle v, w \rangle$$

for all $g \in G$ and $v, w, \in V$, then we say that ρ is a ***unitary*** *representation with respect to $\langle \cdot, \cdot \rangle$.*

Remark 3.14. Note that not every representation is unitary with respect to every inner product, but it turns out that we can construct a new inner product from an old one such that with respect to the new inner product the representation will be unitary.

Theorem 3.15. *Ever representation is a direct sum of irreducible representations.[10]*

3.2 Characters

Characters are a useful tool to help classify representations. More importantly they carry important information about the representation in a more compact form. Character theory was used in the classification of finite simple groups. In this section we state definitions and important results that will aid us in the next section. Each theorem, lemma or definition can be found in [10].

Definition 3.16 (character)**.** *Let $\rho(g)$, $g \in G$ be a matrix representation. Then the **character** of ρ is defined as:*

$$\chi(g) = \operatorname{tr} \rho(g)$$

where tr *is the trace of the matrix.*

Note that elements in the same conjugacy class have the same character value.

Definition 3.17. *Let G be a group. The **character table** of G is an array with rows indexed by the non-equivalent irreducible characters of G and columns indexed by the conjugacy classes.*

Definition 3.18. *Let χ and ψ be any two functions from a group G to the complex numbers. The **inner product** of χ and ψ is defined as :*

$$\langle \chi, \psi \rangle = \frac{1}{|G|} \sum_{g \in G} \chi(g) \overline{\psi(g)}.$$

Theorem 3.19. *Let G be a finite group and ρ a representation of G with associated character χ. Then ρ is irreducible if and only if $\langle \chi, \chi \rangle = 1$.*

Theorem 3.20. *Two representations of a finite group are equivalent if and only if they have the same characters.*

Theorem 3.21. *The number of non-equivalent irreducible representations of the group is equal to the number of conjugacy classes.*

Lemma 3.22. *Let G be a group. Then G is abelian if and only if all the irreducible representations of G have degree 1.*

Lemma 3.23. *Let G be a finite group, then the number of 1-dimensional representations is equal to $|G/[G, G]|$, where $[G, G]$ denotes the commutator subgroup of G.*

3.3 Examples of Representations

One goal of finite group representation theory is to classify all irreducible representations of a group. One can gather some information about the characters and representations that gives you results about the group. As it turns out, all irreducible representations of cyclic groups, abelian groups, dihedral groups and many other groups have been classified. In this section we construct the irreducible representations of the cyclic and dihedral groups. We use most of the results in the previous section to construct these irreducible representations. We will need these families of representations to aid us in proving our main result. These constructions can be found in [7] and [10].

Example 3.24. We find all unitary irreducible representations of $\mathbb{Z}_n$.

From lemma 3.22 above, we know that every irreducible representation of $\mathbb{Z}_n$ is of degree 1. Now, intuitively the elements of the cyclic group of order n are the rotations of an n-gon that leave the n-gon fixed. Now rotations in the complex plane look like $e^{i\theta}$. With a moment's thought, we realize that $e^{\frac{2\pi i}{n}}$ is a rotation of $\frac{2\pi}{n}$ that leaves the n-gon invariant about the origin. We quickly realize that $\langle e^{\frac{2\pi i}{n}} \rangle$ is a cyclic subgroup of $\mathbb{C}^*$, hence isomorphic to $\mathbb{Z}_n$– because any two cyclic groups of order n are isomorphic. Now, let ρ be a function such that $\rho : \mathbb{Z}_n \to \mathbb{C}^*$, where $\rho(1) = e^{\frac{2\pi i}{n}}$, since in an isomorphism a generator must be mapped to a generator. We also know that a generator dictates the map, so $\rho(a) = e^{\frac{2a\pi i}{n}}$ where $a \in \mathbb{Z}_n$. We show that for each $1 \leq j \leq n, \rho_j(a) = e^{\frac{2aj\pi i}{n}}$ is a homomorphism from $\mathbb{Z}_n$ to $\mathbb{C}^*$. Let $a, b \in \mathbb{Z}_n$. Then $\rho_i(a+b) = e^{\frac{2(a+b)j\pi i}{n}} = e^{\frac{2aj\pi i}{n}} \cdot e^{\frac{2bj\pi i}{n}} = \rho(a) \cdot \rho(b)$. We now need to show that these irreducible representations are pairwise distinct to show we have constructed all n representations. This is done by showing $\langle \rho_j(l), \rho_k(l) \rangle = 0$ for $0 \leq k < j \leq n-1$. So $\langle \rho_j(l), \rho_k(l) \rangle = \frac{1}{|G|} \sum_{l=0}^{n-1} e^{\frac{2lj\pi i}{n}} \cdot e^{\frac{-2ak\pi i}{n}} = \frac{1}{|G|} \sum_{l=0}^{n-1} e^{\frac{2l(j-k)\pi i}{n}}$. Let $\xi = e^{\frac{2(j-k)\pi i}{n}}$. Then we get $\frac{1}{|G|} \sum_{l=0}^{n-1} \xi^l = \frac{1}{|G|} \cdot \frac{1-\xi^n}{1-\xi} = 0$, since $1 - \xi \neq 0$ and $\xi \in \langle e^{\frac{2\pi i}{n}} \rangle$. This shows that these representations are inequivalent to each other; ergo, we have n one-dimensional irreducible representations of $\mathbb{Z}_n$. Lastly, these representations are unitary, because $\langle e^{\frac{2\pi iaj}{n}} z_1, e^{\frac{2\pi iaj}{n}} z_2 \rangle_2 = e^{\frac{2\pi iaj}{n}} z_1 \cdot e^{\frac{-2\pi iaj}{n}} \overline{z_2} = e^{\frac{2\pi iaj}{n}} \cdot e^{\frac{-2\pi iaj}{n}} \cdot z_1 \cdot \overline{z_2} = \langle z_1, z_2 \rangle$.

Example 3.25. We find all unitary irreducible representations of D_n.

The dihedral group of order $2n$ can be defined as:

$$D_n := \langle r, s : r^n = s^2 = e, rs = sr^{-1} \rangle = \{e, r, r^2, \ldots, r^{n-1}, s, sr, \ldots, sr^{n-1}\}.$$

Let us find the conjugacy classes for D_n. This will tell us how many irreducible representations it has by theorem 3.21. Our first step is to compute the conjugacy class of a rotation r^i. Conjugating r^i by another rotation, r^j, we get the same rotation r^i. Let us now conjugate r^i by some reflection sr^k:

$$sr^k(r^i)(sr^k)^{-1} = sr^k r^i r^{-k} s = sr^{k+i-k} s = sr^i s = ssr^{-i} = r^{-i}.$$

We can now conclude that the conjugacy class of r^i consists of $\{r^i, r^{-i}\}$. We notice that if n is even, the inverse of $r^{n/2}$ is itself. Thus the conjugacy classes for the rotations, when n is even, breaks up into these class types: $\{e\}, \{r^{n/2}\}, \{r^i, r^{-i}\}$ for $i = 1, \ldots (n-2)/2$. Now if n is odd, the conjugacy classes break up into two types: $\{e\}, \{r^i, r^{-i}\}$ for $i = 1, \ldots, (n-1)/2$.

Let us now compute the conjugacy class of an element of the form sr^k.

If we conjugate by some r^i, we get:

$$r^i(sr^k)r^{-i} = sr^{k-2i}.$$

If we conjugate by some sr^i, we get:

$$sr^i(sr^k)(sr^i)^{-1} = sr^{2i-k}.$$

We readily see that the conjugacy class of an element of the form sr^k consists of $\{sr^{k-2i} : i \in \mathbb{Z}\}$. From the above, we see that sr is conjugate to $sr^3, sr^5 \ldots$ while s is conjugate to $sr^2, sr^4, \ldots$ and these two sets are disjoint if n is even. However, sr is conjugate to sr^{n-1} (via r) so if n is odd, all nontrivial reflections form a single conjugacy class. This information is summarized below.

If n is odd, all of the reflections form a single conjugacy class:

$$\{s, sr, \ldots, sr^{n-1}\}.$$

If n is even then the reflections break into two different classes:

$$\{s, sr^2, \ldots, sr^{n-2}\} \text{ and } \{sr, sr^3, \ldots, sr^{n-1}\}.$$

Let us now count how many conjugacy classes we have. First, let us consider when n is odd. From the analysis above, conjugacy classes of rotations come in pairs.

Since we have $n-1$ rotations and n is odd, we have $(n-1)/2$ conjugacy classes. Now because the identity element is always in its own conjugacy class we have one more conjugacy class. And from our previous calculations the reflections form a single conjugacy class when n is odd. This gives us a total of $(n+3)/2$ conjugacy classes when n is odd.

Let us consider now when n is even. We have again $n-1$ rotations. We have also again that rotations come in pairs when n is even. The only difference in this case is that $r^{n/2}$ is in its own conjugacy classes; hence, we have $(n-2)/2+2$ conjugacy classes of rotations and 2 conjugacy classes of reflections. We have in total $n/2$ +3 conjugacy classes when n is even.

We are now ready to construct our classes of representations for the dihedral group. Let $\xi = e^{\frac{2\pi i}{n}}$. For each integer j with $1 \leq j < n/2$, define

$$R_j = \begin{pmatrix} \xi^j & 0 \\ 0 & \xi^{-j} \end{pmatrix}, \; S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, SR_j = \begin{pmatrix} 0 & \xi^{-j} \\ \xi^j & 0 \end{pmatrix},$$

We can readily see that the following equations hold:

$$R_j^n = S^2 = I, R_jS = SR_j^{-1}.$$

We note that $\langle R_j, S\rangle$ forms a subgroup of $GL(2, \mathbb{C})$

If we define $\rho_j : D_n \to GL(2, \mathbb{C})$ by:

$$\rho_j(r^a) = \begin{pmatrix} \xi^{aj} & 0 \\ 0 & \xi^{-aj} \end{pmatrix},$$

$$\rho_j(sr^a) = \begin{pmatrix} 0 & \xi^{-aj} \\ \xi^{aj} & 0 \end{pmatrix} \; (a, b, \in \mathbb{Z}).$$

we obtain a representation ρ_j of D_n for each j with $1 \leq j < n/2$.

We now show these representations of degree two are irreducible and distinct. We use the inner product method to show this.

Let $1 \leq l \leq j < \frac{n}{2}$, then $\langle \chi(\rho_j), \chi(\rho_l) \rangle = \frac{1}{|D_n|} \sum_{k=0}^{n-1} \chi(\rho_j(r^k))\chi(\rho_l(r^k)) + \sum_{k=0}^{n-1} \chi(\rho_j(sr^k))\chi(\rho_l(sr^k)) = \frac{1}{|D_n|} \sum_{k=0}^{n-1} (\xi^{kj} + \xi^{-kj})(\xi^{kl} + \xi^{-kl}) = \frac{1}{|D_n|} \sum_{k=0}^{n-1} (\xi^{k(j-l)} + \xi^{k(j+l)} + \xi^{-k(j+l)} + \xi^{k(l-j)})$ by our assumption, we get $0 < j+l < n$. therefore, $\xi^{j+l} \neq 1$. $\sum_{k=0}^{n-1} \xi^{k(j+l)} = \frac{\chi^{n(j+l)}-1}{\chi^{(j+l)}-1} = 0$ by the same reasoning $\sum_{k=0}^{n-1} \xi^{-k(j+l)} = 0$. We now get $\langle \chi(\rho_j), \chi(\rho_l) \rangle = \frac{1}{|D_n|} \sum_{k=0}^{n-1} (\xi^{k(j-l)} + \xi^{k(l-j)})$.

If $l = k$ then $\frac{1}{|D_n|} \sum_{k=0}^{n-1} (\xi^{0k} + \xi^{0k}) = 1$ which shows that ρ_j is an irreducible representation for all $1 \leq j < \frac{n}{2}$. We now show all these representations are distinct. If $l \neq k$, then $0 < l - k < n$ and $\sum_{k=0}^{n-1} \xi^{k(j-l)} = \sum_{k=0}^{n-1} \xi^{k(l-j)} = 0$ which shows each ρ_j are distinct for every $1 \leq \rho < n/2$.

We have now constructed distinct irreducible characters χ_j of D_n, one for each j which satisfies $1 \leq j < n/2$.

Let us now find the one-dimensional irreducible representations. From lemma3.23, it is enough to find the commutator subgroup of G.

$$[r^i, r^j] = r^i(r^j)r^{-i}(r^j)^{-1} = e,$$

$$[r^i, sr^j] = r^i(sr^j)r^{-i}(sr^j)^{-1} = (r^2)^i,$$

$$[sr^j, r^i] = (sr^j)r^i(sr^j)^{-1}(r^i)^{-1} = (r^2)^{-i},$$

$$[sr^i, sr^j] = sr^i(sr^j)(sr^i)^{-1}(sr^j)^{-1} = (r^2)^{(j-i)}.$$

Ergo, the commutator subgroup is equal to $\langle r^2 \rangle$.

Now, when n is even we will have four one-dimensional irreducible representations and when n is odd we will have two one-dimensional irreducible representations. The representations for the one-dimensional case are equal to the character of the representation. A final note: Each two-dimensional representation is unitary since each matrix is equal to its own conjugate transpose.

We give the one-dimensional irreducible representation of D_n below. The first table gives the nontrivial irreducible representations when n is even.

χ	r^k	sr^k
χ_1	1	1
χ_2	1	-1
χ_3	$(-1)^k$	$(-1)^k$
χ_4	$(-1)^k$	$(-1)^{k+1}$

This table gives the nontrivial irreducible representations when n is odd.

χ	r^k	sr^k
χ_1	1	-1

CHAPTER 4

Kazhdan constants

This chapter will begin with the definition of the Kazhdan constant followed by some properties relating to the Kazhdan constant. In the next section we compute the Kazhdan constant of tangible abelian groups so the reader can become familiar with the techniques used. We then state known results of computed Kazhdan constants.

4.1 Properties of the Kazhdan Constant

This section consists of the definition of the Kazhdan constant with some necessary properties of the Kazhdan constant. A good amount of the results from the next chapter will refer to these properties to help eliminate redundant cases when computing the Kazhdan constant.

Definition 4.1. *Let G be a finite group, and let $\Gamma \subset G$. Let ρ be a unitary representation of G on some representation space V with G-invariant inner product $\langle \cdot, \cdot \rangle$ where $\| \cdot \|$ is the associated norm. When $\Gamma \neq \phi$, we define*

$$\kappa(G, \Gamma, \rho, \langle \cdot, \cdot \rangle) = \min_{\|v\|=1} \max_{\gamma \in \Gamma} \|\rho(\gamma)v - v\|.$$

Definition 4.2. *Let G be a finite nontrivial group, and let $\Gamma \subset G$. Define*

$$\kappa(G, \Gamma) = \min_{\rho} \{\kappa(G, \Gamma, \rho)\},$$

where the minimum is over all nontrivial irreducible unitary representations of G. The quantity $\kappa(G, \Gamma)$ is called the **Kazhdan constant of the pair** *(G, Γ).*

Lemma 4.3. *If ϕ and ρ are equivalent representations, then $\kappa(G, \Gamma, \phi) = \kappa(G, \Gamma, \rho)$.*

The proof of this can be found in [7].

Remark 4.4. Note that representations in $GL(V)$ are equivalent to representations in $GL(n, \mathbb{C})$ where n is the dimension of the vector space V. Due to this fact, we mostly work with matrix representations to find the Kazhdan constant in the upcoming chapter.

Lemma 4.5. *Let G be a group and $\Gamma_1, \Gamma_2 \subset G$. Suppose that $\Gamma_1 \cup \Gamma_1^{-1} = \Gamma_2 \cup \Gamma_2^{-1}$. Then $\kappa(G, \Gamma_1) = \kappa(G, \Gamma_2)$.*

Proof. It is enough to show that $||\rho(\gamma)v - v|| = ||\rho(\gamma^{-1})v - v||$ since the Kazhdan constant ranges over the same irreducible representations for both norms. Without loss of generality assume ρ is a unitary irreducible representation of G on some representation space V with a G-invariant inner product.

$$||\rho(\gamma)v - v|| = ||\rho(\gamma)v - Iv|| = ||\rho(\gamma)v - \rho(e)v|| = ||\rho(\gamma)v - \rho(\gamma\gamma^{-1})v|| = ||\rho(\gamma)v - \rho(\gamma)\rho(\gamma^{-1})v|| = ||\rho(\gamma)[v - \rho(\gamma^{-1})v]|| = ||v - \rho(\gamma^{-1})v|| = ||\rho(\gamma^{-1})v - v||.$$

Here we use that ρ is unitary with respect to the inner product on V. □

Lemma 4.6. *Let G be a group and $\Gamma_1, \Gamma_2 \subset G$. If there exists $\phi \in Aut(G)$ such that $\phi(\Gamma_1) = \Gamma_2$ then $\kappa(G, \Gamma_1) = \kappa(G, \Gamma_2)$.*

Proof. The crux of the argument is to show that the set $\{\rho_1, \rho_2, \ldots, \rho_n\}$ of all irreducible representations on V has the same cardinality as the set of all irreducible representations of the form $\rho_i \circ \phi$, where ϕ is an automorphism of G. Suppose ρ is a nontrivial irreducible representation on V, and $\phi \in Aut(G)$. Then $\rho \circ \phi$ is a representation on V, since the composition of two homomorphisms is a homomorphism. We need to show that $\rho \circ \phi$ is a nontrivial representation whenever ρ is a nontrivial representation.

Claim 4.7. If ρ is a nontrivial representation from G to $GL(V)$, and $\phi \in Aut(G)$, then $\rho \circ \phi$ is a nontrivial representation from G to $GL(V)$.

Proof. We prove the contrapositive of the statement. Suppose $\rho \circ \phi$ is the trivial representation, then $\rho \circ \phi(g_1) = id_v$, for all $\phi(g_1) \in G$, where id_v is the identity on V and $g_1 \in G$. Since ϕ is onto, for all $g \in G$, there exists a $g_1 \in G$ such that $\phi(g_1) = g$. Therefore, $\rho(g) = id_v$ for all $g \in G$. This proves the claim. $\square$

Claim 4.8. If ρ is a nontrivial irreducible representation from G to $GL(V)$, and $\phi \in Aut(G)$, then $\rho \circ \phi$ is a nontrivial irreducible representation from G to $GL(V)$.

Proof. We prove the contrapositive of the statement. Suppose $\rho \circ \phi$ is reducible. Then there exists a proper nontrivial subspace W of V such that $\rho \circ \phi(g)w \in W$ for all $g \in G$ and $w \in W$. Since ϕ is an automorphism of the group, it is an onto function unto itself. Hence $\rho(g_1)w \in W$ for all $g_1 \in G$ and $w \in W$, where $g_1 = \phi(g)$. This says that ρ is a reducible representation, which proves the claim. $\square$

Let $\phi \in Aut(G)$. $A = \{\rho \mid \rho : G \to GL(V)$ is a nontrivial irreducible representation$\}$ and $B = \{\rho \circ \phi \mid \rho \circ \phi : G \to GL(V)$ is a nontrivial irreducible representation$\}$.

Claim 4.9. $|A| = |B|$.

Proof. Let $T : A \to B$ such that $T(\rho) = \rho \circ \phi$ and $S : B \to A$ such that $S(\sigma) = \sigma \circ \phi^{-1}$. Let $\rho \in A$. Then $S \circ T(\rho) = S(\rho \circ \phi) = \rho$. Let $\rho \circ \phi \in B$. Then $T \circ S(\rho \circ \phi) = T(\rho) = \rho \circ \phi$. This shows that $S = T^{-1}$, which implies that the cardinality of the sets A and B are the same. This proves the claim. $\square$

The lemma now follows from the following equations.

$$\kappa(G,\Gamma_1) = \min_{\rho} \min_{\|v\|=1} \max_{\gamma_1 \in \Gamma_1} \|\rho(\gamma_1)v - v\|$$

$$= \min_{\rho\circ\phi} \min_{\|v\|=1} \max_{\gamma_1 \in \Gamma_1} \|\rho \circ \phi(\gamma_1)v - v\|$$

$$= \min_{\rho} \min_{\|v\|=1} \max_{\gamma_2 \in \Gamma_2} \|\rho(\gamma_2)v - v\| = \kappa(G,\Gamma_2). \qquad \square$$

Remark 4.10. Note that the Kazhdan constant is non-negative, therefore bounded below. Now a follow-up question to this would be "Is the set of all Kazhdan constants bounded above?" The next proposition answers this question.

Proposition 4.11. *Let G be a finite nontrivial group, and $\Gamma \subset G$. Then $\kappa(G,\Gamma) \leq 2$.*

Proof. Let ρ be a unitary irreducible representation on a vector space V,then:

$$\kappa(G,\Gamma) \leq \max_{\gamma\in\Gamma} \|\rho(\gamma)v - v\| \leq \|\rho(\gamma)v\| + \|v\| = 2. \qquad \square$$

The following theorem is what makes the Kazhdan constant relevant to graph theory.

Theorem 4.12. *Let d be a positive integer. Let (G_n) be sequence of groups with $|G_n| \to \infty$. For each n, let $\Gamma_n \subseteq_s G_n$ such that $|\Gamma_n| = d$. Then $(Cay(G_n,\Gamma_n))$ is an expander family if and only if there exists $\epsilon > 0$ such that $\kappa(G_n,\Gamma_n) \geq \epsilon$ for all n.[7]*

Remark 4.13. So far we have some pretty neat results about the Kazhdan constant but never really mention if the Kazhdan constant exists. We show existence in the next theorem, but we need a lemma first.

Lemma 4.14. *If V is a normed vector space, then$f : V \to \mathbb{R}$ defined by $f(v) = \|v\|$ for all $v \in V$ is continuous.*

Proof. Let $\epsilon > 0$, $\delta = \epsilon$ and assume $\|v - v_0\| < \delta$, then $|f(v) - f(v_0)| = |\|v\| - \|v_0\|| \leq \|v - v_0\| \leq \delta = \epsilon$ which proves the result. $\square$

Theorem 4.15. *The Kazhdan constant always exists, for finite groups.*

Proof. Let G be a finite group, and let $\Gamma \subset G$. Without loss of generality assume ρ is a nontrivial unitary representation of G on some representation space V with G-invariant inner product $\langle \cdot, \cdot \rangle$. We have that

$$\kappa(G, \Gamma, \rho, \langle \cdot, \cdot \rangle) = \min_{\|v\|=1} \max_{\gamma \in \Gamma} \|\rho(\gamma)v - v\|.$$

From lemma 4.14 above we know $\| \cdot \|$ is a continuous function on V. Since $|\Gamma| \leq |G|$ and G is finite, Γ is finite. Therefore, $\max \|\rho(\gamma)v - v\|$ is a continuous function, but our domain is compact so it attains a minimum. Hence $\kappa(G, \Gamma, \rho, \langle \cdot, \cdot \rangle)$ exists for each ρ, but we only have finitely many irreducible representations up to equivalence so $\min_\rho\{\kappa(G, \Gamma, \rho)\}$ also exists, but

$$\kappa(G, \Gamma) = \min_{\rho}\{\kappa(G, \Gamma, \rho)\}.$$

This proves the statement. $\square$

4.2 Examples and Known Results

In this section we compute Kazhdan constants of abelian groups in order to get the feel for computing Kazhdan constants. We define the circular norm which will aid us in computing Kazhdan constants in the next chapter. We then state known results on Kazhdan constants. We first need a preliminary lemma.

Lemma 4.16. *Whenever* $0 \leq \theta \leq 2\pi$, *we have* $|e^{i\theta} - 1| = 2 \sin \frac{\theta}{2}$.

Proof. Let $\theta \in [0, 2\pi]$, then $|e^{i\theta} - 1|^2 = (\cos(\theta) - 1)^2 + \sin^2(\theta) = 2(1 - \cos(\theta)) =$

$4\sin^2(\frac{\theta}{2})$. Taking square roots, we get:

$$|e^{i\theta} - 1| = 2\sin\frac{\theta}{2}$$

since $\sin(\frac{\theta}{2}) \geq 0$ for $0 \leq \frac{\theta}{2} \leq \pi$. $\square$

Example 4.17. We will compute $\kappa(\mathbb{Z}_4, \{1,3\})$.

From lemma 4.5 we know $\kappa(\mathbb{Z}_4, \{1\}) = \kappa(\mathbb{Z}_4, \{1,3\})$.

Let us first recall the one-dimensional, nontrivial, irreducible representations of $\mathbb{Z}_4$:

$$\rho_j(\gamma) = e^{\frac{2\pi\gamma j i}{4}}, \text{ such that } 1 \leq j \leq 3 \text{ and } \gamma \in \mathbb{Z}_4.$$

By the definition of the Kazhdan constant, we have:

$$\kappa(G,\Gamma) = \min_{\rho_j} \max_{\gamma\in\Gamma} \|\rho_j(\gamma) - 1\| = \min_{\rho_j} \max_{1\in\{1\}} \|e^{\frac{\pi j i}{2}} - 1\|$$

$$= \min\{|e^{\frac{\pi i}{2}} - 1|, |e^{\frac{2\pi i}{2}} - 1|, |e^{\frac{3\pi i}{2}} - 1|\} = |e^{\frac{\pi i}{2}} - 1| = 2\sin\frac{\pi}{4} = \sqrt{2}.$$

Definition 4.18. *Let n be a positive integer. Let $LR(k)$ be the least residue of k mod n i.e $LR(k) \equiv k \pmod{n}$, and $0 \leq LR(k) < n$.The* ***circular norm of k, CN(k),*** *is defined to be equal to* $\min\{LR(k), LR(-k)\}$.

Example 4.19. We find $\kappa(K4G, \{a,b\})$. To accomplish this task we must first find all irreducible representations of this group. Readers can show that the table below completes the list of all irreducible representations for the Klein group.

ρ	e	a	b	ab
ρ_0	1	1	1	1
ρ_1	1	-1	1	-1
ρ_2	1	-1	-1	1
ρ_3	1	1	-1	-1

Now for each representation $\rho_i(a)$ or $\rho_i(b)$ is equal to -1. Therefore,

$$\max_{\gamma \in \{a,b\}} \|\rho_i(\gamma)v - v\| = 2$$

from which we can now conclude that $\kappa(K4G, \{a,b\}) = 2$.

The examples we gave were very modest. Only in a few cases have the Kazhdan constants been computed. We state some known results below. Note that this list is not exhaustive. The first results were discovered by [2].

Theorem 4.20.

(1) $\kappa(\mathbb{Z}_n, \{1\}) = 2\sin\frac{\pi}{n}$,

(2) $\kappa(D_n, \{s,r\}) = 2\sin(\frac{\pi}{2n})$,

(3) $\kappa(S_n, \{(1,2), \ldots, (n-1,n)\}) = \sqrt{\frac{24}{n^3-n}}$.

In Derbidge's master's thesis [3] he shows the following result.

Theorem 4.21. *Let* $\Gamma = \{1, 3, \ldots, 2n-1\}$ *in* $\mathbb{Z}_{2n}$, *then*

$$\kappa(\mathbb{Z}_{2n}, \Gamma) = \begin{cases} \sqrt{2} & \textit{if } n \textit{ is even,} \\ 2\cos(\frac{\pi}{2p}) & \textit{if } n \textit{ is odd, where } p \textit{ is the smallest odd prime dividing } n. \end{cases}$$

We will prove (1) from theorem 4.20 in the next section, since it is relevant to this thesis.

CHAPTER 5

Main Results

Now that we have all the necessary background knowledge, we can move forward and discuss new results involving Kazhdan constants. When we started studying the Kazhdan constant, we had strong hopes that it would be a graph invariant. Hence the goal for this research was to show that the Kazhdan constant was an invariant under isomorphic Cayley graphs. As it turns out, this is not the case and we need a stronger condition for the Kazhdan constant to be an invariant. In the first section, we give an example of infinitely many pairs, each time involving distinct groups, where both the Cayley graphs and the Kazhdan constants are the same. In the second section we show that Cayley isomorphism is a sufficient condition for our conjecture to hold but not a necessary condition. The last section will be dedicated to conjectures and our thoughts on the subject.

5.1 Isomorphic Cayley graphs with different groups and equal Kazhdan constants

This section is devoted to proving that our conjecture about the Kazhdan constant holds infinitely many times. We also state some corollaries of the main theorem. With all the background information already handled, we are now ready to attack the main result.

Theorem 5.1. *There exist infinitely many graphs such that:*

(1) $Cay(G_1, \Gamma_1) \cong Cay(G_2, \Gamma_2)$, *and*

(2) $\kappa(G_1, \Gamma_1) = \kappa(G_2, \Gamma_2)$,

where G_1, G_2 *are non-isomorphic groups and* Γ_1, Γ_2 *are subsets of* G_1, G_2 *respectively.*

Proof. The proof we give is constructive.

Let G_1 be the dihedral group D_n of order $2n$, with $n \geq 3$.

$$D_n = \langle r, s : r^n = s^2 = e, s^{-1}rs = r^{-1} \rangle,$$

and $X = Cay(D_n, \{s, sr\})$.

Let G_2 be the cyclic group of order $2n$ and $Y = Cay(\mathbb{Z}_{2n}, \{1, 2n-1\})$. Note that since X and Y are both 2-regular connected graphs, they must be cycle graphs of order 2n, hence they are the same graph isomorphic. We have thus proved (1).

To accomplish (2) we first work with $\kappa(\mathbb{Z}_{2n}, \{1, 2n-1\})$.

From the lemma 4.5 above we know $\kappa(\mathbb{Z}_{2n}, \{1\}) = \kappa(\mathbb{Z}_{2n}, \{1, 2n-1\})$.

Let us first recall the one-dimensional, nontrivial, irreducible representations of $\mathbb{Z}_k$:

$$\rho_j : \mathbb{Z}_{2n} \to \mathbb{C}^* \text{ defined by}$$

$$\rho_j(\gamma) = e^{\frac{2\pi\gamma ji}{2n}}, \text{ such that } 1 \leq j \leq 2n-1 \text{ and } \gamma \in \mathbb{Z}_{2n}.$$

By the definition of the Kazhdan constant, we have:

$$\kappa(\mathbb{Z}_{2n}, \{1, 2n-1\}) = \min_{\rho_j} \max_{\gamma \in \Gamma} \|\rho_j(\gamma) - 1\| = \min_{\rho_j} \max_{1 \in \{1\}} \|e^{\frac{2\pi ji}{2n}} - 1\|$$

$$= \min\{|e^{\frac{2\pi i}{2n}} - 1|, |e^{\frac{2(2)\pi i}{2n}} - 1|, \ldots, |e^{\frac{2(k-1)\pi i}{2n}} - 1|\} = |e^{\frac{2\pi i}{2n}} - 1| = 2\sin\frac{\pi}{2n}.$$

We now calculate $\kappa(D_n, \{s, sr\})$.

Recall that for a 2-dimensional representation of the dihedral group, s and sr get mapped to:

$$\rho_j : D_n \to GL(2, \mathbb{C})$$

$$\rho_j(s) = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \rho_j(sr) = \begin{pmatrix} 0 & \xi^{-j} \\ \xi^j & 0 \end{pmatrix}.$$

Let $v = (a, b)^T, |a|^2 + |b|^2 = 1, a, b \in \mathbb{C}$, then $\|\rho_j(s)v - v\| = \|(b-a, a-b)\| = \sqrt{2}|b-a|$.

Similarly, $\|\rho_j(sr)v - v\| = \|(b\xi^j - a, a\xi^{-j} - b)\| = \sqrt{|\xi^j b - a|^2 + |\xi^{-j}a - b|^2} = \sqrt{2}|\xi^j b - a|$,

$$\text{since } |\xi^j b - a| = |\xi^j(b - a\xi^{-j})| = |\xi^j||b - \xi^{-j}a| = |\xi^{-j}a - b|.$$

Now the angle from a to $a\xi^{-j}$ is $2j\pi/n$. Apply a rotation in order for the imaginary axis to bisect the angle from a to $a\xi^{-j}$. Therefore, the angle from the x-axis to $a\xi^{-j}$ is $\pi/2 - j\pi/n = (n\pi - 2j\pi)/2n$. In other words, $a = re^{[(n\pi+2j\pi)/2n]i}$, where $1 \leq j < n/2$ for some positive real number r. Refer to Figure 5.1.

We can now see if $\gamma \in \Gamma$, then

$$\max \|\rho_j(\gamma)v - v\| = \max \begin{cases} \sqrt{2}|b - \xi^{-j}a| & \text{Re }(b) \leq 0, \\ \sqrt{2}|b - a| & \text{Re }(b) \geq 0. \end{cases}$$

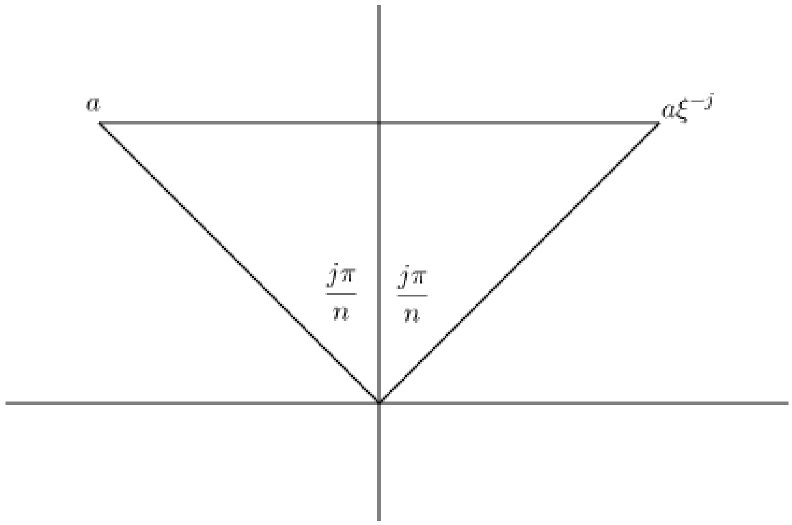


Figure 5.1: Justification for θ

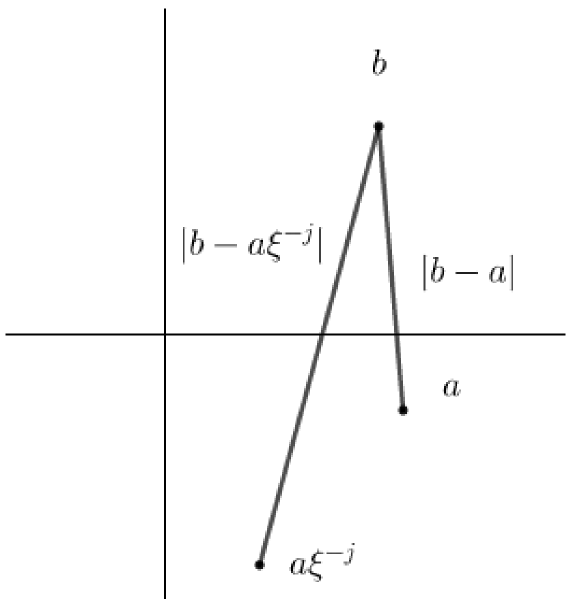


Figure 5.2: Visualization of argument

Refer to Figure 5.2 and 5.3. Without loss of generality, assume Re $b \geq 0$. We know $a = re^{i\theta}$ where $\theta = \frac{n\pi+2j\pi}{2n}$ and $1 \leq j < n/2$. Let $b = x + iy$ where $x, y \in \mathbb{R}$ and $x^2 + y^2 + r^2 = 1$. We seek to minimize $|a - b|^2$.

Let $|a - b|^2 := f(r, x, y) = |re^{i\theta} - (x + iy)|^2$. Then $f(r, x, y) = |(r\cos\theta - x) +$

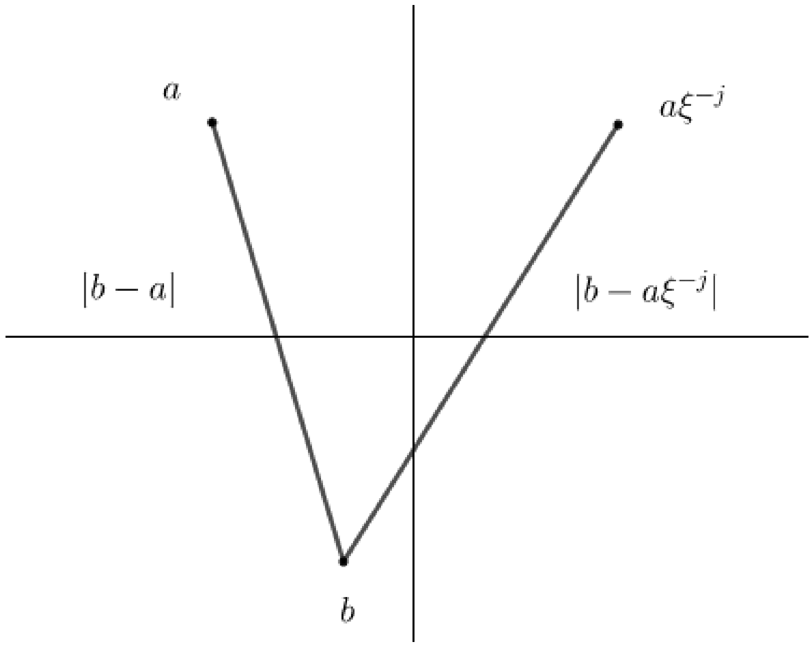


Figure 5.3: Visualization of argument rotated

$(r\sin\theta - y)i|^2 = (r\cos\theta - x)^2 + (r\sin\theta - y)^2 = r^2\cos^2\theta - 2rx\cos\theta + x^2 + r^2\sin^2\theta - 2ry\sin\theta + y^2 = 1 - 2rx\cos\theta - 2ry\sin\theta$. Let $g(r,x,y) = r^2 + x^2 + y^2 - 1$. If we invoke the Lagrange multiplier method we get the following equations, for some real number λ:

$$x\cos\theta + y\sin\theta = -r\lambda \tag{5.1}$$

$$r\cos\theta = -x\lambda \tag{5.2}$$

$$r\sin\theta = -y\lambda. \tag{5.3}$$

Multiplying the first equation by r, the second equation by $-x$, the third equation by $-y$ then adding each each equation we get:

$$0 = -r^2\lambda + x^2\lambda + y^2\lambda.$$

Adding $2r^2\lambda$ to both sides and using our relation of the variables r, x, y yields the new equation:

$$2r^2\lambda = \lambda.$$

We have two cases to consider: $\lambda = 0$ and $r = \frac{\sqrt{2}}{2}$.

If $\lambda = 0$, then $r\cos\theta = 0$, so $r = 0$ or $\cos\theta = 0$. Note that $\frac{\pi}{2} < \theta < \pi$, which means $\cos\theta$ is never zero. If $r = 0$ then $a = 0$ and $|b - a|^2 = 1$.

Let $r = \frac{\sqrt{2}}{2}$. Substituting this value for r above and multiplying row 2 by $-y$, row 3 by x and adding these two equations gives us: $y = x\tan\theta$. Substituting r and y into $g(r, x, y)$ we get $x = \frac{\sqrt{2}}{2}|\cos\theta|$. Recall $f(r, x, y) = 1 - 2rx\cos\theta - 2ry\sin\theta$. So $f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}|\cos\theta|, \frac{\sqrt{2}}{2}|\cos\theta|\tan\theta) = 1 - |\cos\theta|\cos\theta - |\cos\theta|\tan\theta\sin\theta$.

We noticed from our choice of θ, $\cos\theta$ will be less than zero, therefore

$f = 1 + \cos^2\theta + \sin^2\theta = 2$.

From the Lagrange multiplier technique, we know our minimum should be less than $\min\{1, 2\}$. We now find the minimum on the boundary, that is when $x = 0$.

Let us consider the function $f(r, 0, y) = 1 - 2ry\sin\theta$ where $r^2 + y^2 = 1$. Let $g(r, 0, y) = r^2 + y^2$. Then we get the following equations for some real γ value:

$$y\sin\theta = -r\lambda$$

$$r\sin\theta = -y\lambda.$$

Multiplying the first equation by r and the second equation by y then adding the two equations yields $r = \frac{\sqrt{2}}{2}$. Solving for y gives $y = \pm\frac{\sqrt{2}}{2}$.

$$f(\frac{\sqrt{2}}{2}, \frac{\sqrt{2}}{2}) = 1 - \sin\theta.$$

$$f(\frac{\sqrt{2}}{2}, -\frac{\sqrt{2}}{2}) = 1 + \sin\theta.$$

Our minimum should now be less than $\min\{1 - \sin\theta, 1 + \sin\theta, 1, 2\}$, and it is clear that the minimum is $1 - \sin\theta$. Therefore $\min|b - a| = \sqrt{1 - \sin\theta}$.

We now find the minimum of j in θ. Recall $\theta = \frac{\pi}{2} + \frac{j\pi}{n}$. Since $\sin\theta$ is decreasing on the interval $(\frac{\pi}{2}, \pi)$ therefore our minimum occurs when $j = 1$, that is, when $\theta = \frac{\pi}{2} + \frac{\pi}{n}$. Therefore, the minimum Kazhdan constant for the two-dimensional representations is

$$\kappa(D_n, \{s, sr\}, \rho_1) = \sqrt{2(1 - \cos\frac{\pi}{n})}.$$

Recall the one-dimensional irreducible representations of D_n. The first table gives the nontrivial irreducible representations when n is even.

χ	r^k	sr^k
χ_1	1	-1
χ_2	$(-1)^k$	$(-1)^k$
χ_3	$(-1)^k$	$(-1)^{k+1}$

This table gives the nontrivial irreducible representations when n is odd.

χ	r^k	sr^k
χ_1	1	-1

Note that $\chi_i(s) = -1$ or $\chi_i(sr) = -1$ when n is even or odd and $1 \leq i \leq 3$.

The minimum Kazhdan constant for the one-dimensional representations is :

$$\kappa(D_n, \{s, sr\}, \chi_k) = 2.$$

Therefore, $\kappa(D_n, \{s, sr\}) = \sqrt{2(1 - \cos\frac{\pi}{n})}$.

The double angle formula then shows $\kappa(D_n, \{s, sr\}) = \kappa(\mathbb{Z}_{2n}, \{1, 2n - 1\})$ $\square$

Corollary 5.2. *Let G be a finite group that has order greater than 4 and $\Gamma \Subset_s G$. If $Cay(G, \Gamma)$ is a cycle graph then $\kappa(G, \Gamma) = 2\sin\frac{\pi}{n}$.*

Proof. So it turns out, up to Cayley isomorphism, only the pair $(\mathbb{Z}_n, \{1, n-1\})$ and $(D_n, \{s, sr\})$ yield cycle graphs. But by theorem 5.1 we know the Kazhdan constant associated to these pairs are equal: $2\sin(\frac{\pi}{n})$. □

Corollary 5.3. *Cycle graphs do not form an expander family.*

Proof. By theorem 5.1 and 5.2, we know that we just need to consider $\kappa(\mathbb{Z}_n, \{1, n-1\})$. But this Kazhdan constant goes to zero as n approaches infinity,

$$\lim_{n\to\infty} \kappa(\mathbb{Z}_n, \{1, n-1\}) = \lim_{n\to\infty} 2\sin(\frac{\pi}{n}) = 0.$$

Because the Kazhdan constant is not bounded away from zero, the associated Cayley graphs of the pair $(\mathbb{Z}_n, \{1, n-1\})$ does not form an expander family. □

Remark 5.4. Remember what the goal of our research was: to determine if the Kazhdan constant was a Cayley graph invariant. As it turns out, combining Examples 4.17 and 4.19 give a counterexample for our conjecture of the Kazhdan constant being a Cayley graph invariant. The question we ask ourselves now is "If we restrict ourselves to only cyclic groups,would the Kazhdan constant be a Cayley graph invariant?" We investigate this idea in the following section.

5.2 Isomorphic Cayley graphs with the same group give equal Kazhdan constants

We gave a remark at the end of section 5.1 stating that the Kazhdan constant is not a graph invariant. In this section we show Cayley isomorphism is enough to show the Kazhdan constant is equal, and the Cayley graphs are isomorphic . The main theorem of this section generalizes a counterexample from [4] disproving Àdàm's conjecture. The theorem below shows that being Cayley isomorphic is not a necessary condition for our conjecture to hold for a family of cyclic groups. With this result proven, we are left to wonder whether being cyclic groups and having isomorphic Cayley graphs makes the Kazhdan constant a Cayley graph invariant.

Theorem 5.5. *There exist infinitely many pairs such that :*

(1) $Cay(G_1, \Gamma_1) \cong Cay(G_2, \Gamma_2)$*, and*

(2) $Cay(G_1, \Gamma_1)$ *is not Cayley isomorphic to* $Cay(G_2, \Gamma_2)$*, and*

(3) $\kappa(G_1, \Gamma_1) = \kappa(G_2, \Gamma_2)$,

where G_1, G_2 *are finite cyclic groups, and* Γ_1, Γ_2 *are subsets of* G_1, G_2 *respectively.*

Proof. The proof is constructive. Let $G_1{=}G_2{=}\mathbb{Z}_n$, where $n \geq 16$ and $8|n$.

Let $\Gamma_1 = \{1, 2, \frac{n}{2} - 1, \frac{n}{2} + 1, n - 2, n - 1\}$ and $\Gamma_2 = \{2, \frac{3n}{4} - 1, \frac{n}{4} - 1, \frac{n}{4} + 1, \frac{3n}{4} + 1, n - 2\}$.

We will start off by showing $Cay(G_1, \Gamma_1) \cong Cay(G_2, \Gamma_2)$.

Consider the mapping $f : \mathbb{Z}_n \to \mathbb{Z}_n$

$$f(i) = \begin{cases} i & \text{if } i \text{ is even,} \\ i + \frac{n}{4} & \text{otherwise.} \end{cases}$$

We first note that this mapping is well-defined, hence a function.

Next we show that f is bijective. It is enough to show that f has an inverse function.

Consider the function $g : \mathbb{Z}_n \to \mathbb{Z}_n$

$$g(i) = \begin{cases} i & \text{if } i \text{ is even,} \\ i - \frac{n}{4} & \text{otherwise.} \end{cases}$$

We claim that g is the inverse of f. We first consider $f(g(i))$. Let i be even. Then $g(i) = i$ so $f(g(i)) = f(i) = i$. Assume now that i is odd. Then $g(i) = i - \frac{n}{4}$ so $f(i - \frac{n}{4}) = i - \frac{n}{4} + \frac{n}{4} = i$.

We next consider $g(f(i))$. Let i be even. Then $f(i) = i$ so $g(f(i)) = g(i) = i$. Assume now that i is odd, then $f(i) = i + \frac{n}{4}$. Therefore, $g(i + \frac{n}{4}) = i + \frac{n}{4} - \frac{n}{4} = i$. Hence, $g = f^{-1}$.

We now show the function f is a graph automorphism. Assume i and j are adjacent i.e. $i - j \in \Gamma_1$. Consider the following cases.

(1) Suppose i and j are both even. Then $f(i) - f(j) = i - j \in \Gamma_2$ since each set has the same even elements and $i - j$ is even.

(2) Suppose i is odd and j even. Then $i - j$ is odd. Now $f(i) - f(j) = i - j + \frac{n}{4}$.

Claim 5.6. If γ is an odd element of Γ_1, then $\gamma + \frac{n}{4} \in \Gamma_2$.

Proof. If we assume γ is an odd element in Γ_1 then we only have four cases to consider.

(a) If $\gamma = \frac{n}{2} - 1$ then, $\frac{n}{2} - 1 + \frac{n}{4} = \frac{3n-4}{4} \in \Gamma_2$.

(b) If $\gamma = \frac{n}{2} + 1$ then, $\frac{n}{2} + 1 + \frac{n}{4} = \frac{3n+4}{4} \in \Gamma_2$.

(c) If $\gamma = 1$ then, $1 + \frac{n}{4} = \frac{n+4}{4} \in \Gamma_2$.

(d) If $\gamma = n - 1$ then, $n - 1 + \frac{n}{4} = -1 + \frac{n}{4} = \frac{n-4}{4} \in \Gamma_2$.

These cases complete the proof. $\square$

Hence, $f(i) - f(j) \in \Gamma_2$ so $f(i)$ is adjacent to $f(j)$.

(3) Suppose i is even and j odd. Then $i - j$ is odd. Because we are working with symmetric subsets $-(i-j) \in \Gamma_1$. Now $f(i) - f(j) = i - j - \frac{n}{4} = -[-(i-j) + \frac{n}{4}]$, but since $-(i-j) \in \Gamma_1$ and is odd, we know $[-(i-j) + \frac{n}{4}] \in \Gamma_2$ by the argument in (2). Appealing to the fact that Γ_2 is a symmetric subset again, we get that $-[-(i-j) + \frac{n}{4}] \in \Gamma_2$ so $f(i)$ is adjacent to $f(j)$.

(4) Suppose i and j are both odd. Then $i - j$ is even. Now $f(i) - f(j) = i + \frac{n}{4} - (j + \frac{n}{4}) = i - j \in \Gamma_2$ since both sets have the same even elements in them. Hence, $f(i)$ and $f(j)$ are adjacent. Ergo, f is a graph isomorphism.

We now show that $(\mathbb{Z}_n, \Gamma_1)$ is not Cayley isomorphic to $(\mathbb{Z}_n, \Gamma_2)$. Assume to the contrary that $(\mathbb{Z}_n, \Gamma_1)$ is Cayley isomorphic to $(\mathbb{Z}_n, \Gamma_2)$. Then there must exist a $\phi \in Aut(\mathbb{Z}_n)$ such that $\phi(\Gamma_1) = \Gamma_2$. Recall that all automorphisms of the group $\mathbb{Z}_n$ look like $f(x) = ax$ where $\gcd(a, n) = 1$. Since $1 \in \Gamma_1$ $f(1) = a$ and also $f(1) \in \Gamma_2$. Therefore, it must be the case that $f(1)$ maps to a generator of $\mathbb{Z}_n$ in Γ_2. We know that the subgroup $\langle 2 \rangle$ does not generate $\mathbb{Z}_n$. Hence, it is enough to consider only four cases for f:

(1) $f_1(k) = \frac{3n+4}{4}k$

(2) $f_2(k) = \frac{n+4}{4}k$

(3) $f_3(k) = \frac{3n-4}{4}k$

(4) $f_4(k) = \frac{n-4}{4}k$

Before we consider these four cases we look at how the automorphism, f_i, acts on the element $2 \in \Gamma_1$. Let γ be an odd element in Γ_2 and recall that $|\gamma| = \frac{n}{\gcd(n,\gamma)}$. Note that the order of γ depends on $\gcd(n,\gamma)$, and $\gcd(n,\gamma)$ is always odd. Hence $|\gamma| \neq \frac{n}{2}$, but $|2| = \frac{n}{2}$ and because f_i is an automorphism of $\mathbb{Z}_n$, $f(2) \neq \gamma$. So it must be the case that $f_i(2) = 2$ or $f_i(2) = n-2$ where $1 \leq i \leq 4$. We consider these subcases below.

Case 1: Consider the function $f_1(k) = \frac{3n+4}{4}k$. Then

$f_1(2) = (\frac{3n+4}{4})2 = 2(n - \frac{n}{4} + 1) = -\frac{n}{2} + 2$

Subcase 1: $f_1(2) = 2 = -\frac{n}{2} + 2$ then $\frac{n}{2} \equiv 0 \pmod{n}$, which is a contradiction.

Subcase 2: $f_1(2) = n - 2 = -\frac{n}{2} + 2$ then $\frac{n}{2} \equiv 4 \pmod{n}$ so $8 \equiv 0 \pmod{n}$ which is a contradiction because $n \geq 16$.

From the results of subcase 1 and 2, we conclude that $f_1(2) \notin \Gamma_2$.

Case 2: Consider the function $f_2(k) = \frac{n+4}{4}k$. Then $f_2(2) = (\frac{n+4}{4})2 = \frac{n}{2} + 2$.

Subcase 1: $f_2(2) = 2 = \frac{n}{2} + 2$. Then $\frac{n}{2} \equiv 0 \pmod{n}$,which is a contradiction from the same reason as the previous case.

Subcase 2: $f_2(2) = n - 2 = \frac{n}{2} + 2$. Then $0 \equiv -8 \pmod{n}$, which is a contradiction.

Hence, $f_2(2) \notin \Gamma_2$.

Case 3: Consider the function $f_3(k) = \frac{3n-4}{4}k$.

Then $f_3(2) = (\frac{3n-4}{4})2 = -(\frac{n}{2} + 2)$, but by Case 2: $\frac{n}{2} + 2 \notin \Gamma_2$ and because Γ_2

is symmetric $f_3(2) \notin \Gamma_2$.

Case 4: Consider the function $f_4(k) = \frac{n-4}{4}k$.

Then $f_4(2) = (\frac{n-4}{4})2 = \frac{n}{2} - 2 = -(-\frac{n}{2} + 2)$. By similar reasoning as in case 3 we can conclude $f_4(2) \notin \Gamma_2$.

Thus these $(\mathbb{Z}_n, \Gamma_1)$ and $(\mathbb{Z}_n, \Gamma_2)$ are not Cayley isomorphic.

We now show $\kappa(\mathbb{Z}_n, \Gamma_1) = \kappa(\mathbb{Z}_n, \Gamma_2)$

Let us recall the one-dimensional, nontrivial, irreducible representations of $\mathbb{Z}_n$:

$$\rho_j(\gamma) = e^{\frac{2\pi\gamma ji}{n}}, \text{ such that } 1 \leq j \leq n-1 \text{ and } \gamma \in \mathbb{Z}_n.$$

Let $\Gamma = \{1, 2, \frac{n}{2} - 1\}$. Then we know by lemma 4.5 $\kappa(G, \Gamma) = \kappa(G, \Gamma_1)$. By the definition of the Kazhdan constant, we have:

$$\kappa(\mathbb{Z}_n, \Gamma) = \min_{\rho_j} \max_{\gamma\in\Gamma} \|\rho_j(\gamma) - 1\|.$$

Now, intuitively, $\|\rho_j(\gamma) - 1\| = \|e^{\frac{2\pi\gamma ji}{n}} - 1\|$ tells us the distance from a certain point on the unit circle to 1. Since we seek to find the maximum of this distance, where γ runs over Γ, we need to look for values of j that $e^{\frac{2\pi\gamma ji}{n}}$ is the minimal distance to $e^{\pi i}$; in other words, we want γj to be near $\frac{n}{2}$. To make our lives easier, it is enough to only consider values from $1 \leq j \leq \frac{n}{2}$, since

$$\|e^{\frac{2\pi\gamma ji}{n}} - 1\| = \|e^{\frac{2\pi\gamma(n-j)i}{n}} - 1\|.$$

We are now ready to find the Kazhdan constant. We can now think of this problem as finding the minimum of the maximum circular norms. The table below summarizes the circular norm for each element of Γ_1.

j	1	2	$\frac{n}{2}-1$	max
1	1	2	$\frac{n}{2}-1$	$\frac{n}{2}-1$
2	2	4	-2	4
3	3	6	$\frac{n}{2}-3$	$\frac{n}{2}-3$
4	4	8	1	8
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.

Note that when $5 \leq j \leq \frac{n}{2}$ the maximum is at least 4. Now the minimum of the set of all maxima is 4, hence

$$\kappa(\mathbb{Z}_n, \Gamma_1) = \kappa(\mathbb{Z}_n, \Gamma) = 2\sin(\tfrac{4\pi}{n}).$$

Next we compute $\kappa(\mathbb{Z}_n, \Gamma_2)$.

Let $\Gamma_3 = \{2, \frac{n}{4}-1, \frac{n}{4}+1\}$. Again, from lemma 4.5, we know $\kappa(\mathbb{Z}_n, \Gamma_2) = \kappa(\mathbb{Z}_n, \Gamma_3)$. The argument to compute this Kazhdan constant is similar to how we found the Kazhdan constant for the pair $(\mathbb{Z}_n, \Gamma_1)$. The table below summarizes the circular norm for each element of Γ_2.

j	2	$\frac{n}{4}-1$	$\frac{n}{4}+1$	max
1	2	$\frac{n}{4}-1$	$\frac{n}{4}+1$	$\frac{n}{4}+1$
2	4	$\frac{n}{2}-2$	$\frac{n}{2}+2$	$\frac{n}{2}+2$
3	6	$\frac{3n}{4}-3$	$\frac{3n}{4}+3$	$\frac{3n}{4}-3$
.	.	.	.	.
.	.	.	.	.
.	.	.	.	.
$\frac{n}{2}-2$	$n-4$	-2	2	$n-4$
$\frac{n}{2}-1$	-2	$\frac{n}{4}+1$	$\frac{n}{4}-1$	$\frac{n}{4}+1$
$\frac{n}{2}$	0	$\frac{n}{2}$	$\frac{n}{2}$	$\frac{n}{2}$

Now the minimum of the set of all maxima is 4, hence $\kappa(\mathbb{Z}_n, \Gamma_3) = 2\sin(\frac{4\pi}{n})$, but $\kappa(\mathbb{Z}_n, \Gamma_2) = \kappa(\mathbb{Z}_n, \Gamma_3) = 2\sin(\frac{4\pi}{n})$.

Hence, $\kappa(\mathbb{Z}_n, \Gamma_2) = \kappa(\mathbb{Z}_n, \Gamma_1)$. □

Remark 5.7. I created an algorithm in C++ in order to calculate formulas for the Kazhdan constant for this type of cyclic group. I have placed the code in the Appendix if the reader is interested. It is possible to generalize this code to find the Kazhdan constant constant for cyclic groups of even order. If you want to find the Kazhdan constant for an odd order cyclic group a little work needs to be done to the logic of the program.

5.3 Conjectures Regarding the Kazhdan Constant

In this section we state conjectures that we hope to investigate in the future.

Conjecture 5.8. Let G a finite cyclic group of order $9n$ where n is greater than 2. Let $\Gamma_i = \{\pm 1, \pm(3n+1), \pm(6n+1), \pm 3(in+1)\}, i = 0, 1, 2$. Then $Cay(\mathbb{Z}_{9n}, \Gamma_0) \cong Cay(\mathbb{Z}_{9n}, \Gamma_i)$ if 3 does not divide $im + 1$. Moreover, $\kappa(\mathbb{Z}_{9n}, \Gamma_0) = \kappa(\mathbb{Z}_{9n}, \Gamma_i)$.

Conjecture 5.9. Let G be a finite cyclic group of order n and $\Gamma_1, \Gamma_2 \subseteq_s G$. If $Cay(\mathbb{Z}_n, \Gamma_1) \cong Cay(\mathbb{Z}_n, \Gamma_2)$ then $\kappa(G, \Gamma_1) = \kappa(G, \Gamma_2)$.

Conjecture 5.10. Let G be a finite group. Then $\kappa(G, \Gamma) = 2$, if and only if $|G| = 2^k$, for some positive integer k.

Question 1. Let $\alpha \in [0, 2] \cap \mathbb{Q}$. Then there exists a group G and $\Gamma \subset G$ such that $\kappa(G, \Gamma) = \alpha$.

A more ambitious conjecture would be as followed.

Question 2. Let $\alpha \in [0, 2]$. Then there exists a group G and $\Gamma \subset G$ such that $\kappa(G, \Gamma) = \alpha$.

Remark 5.11. To even attempt to attack this conjecture we can no longer be restricted to finite groups.

REFERENCES

[1] A. Ádám, *Research problem 2-10*, J. Combin. Theory 2 (1967), 393.

[2] R. Bacher and P. de la Harpe, *Exact values of Kazhdan constants for some finite groups*, J. Algebra, **163** (1994), 495—515.

[3] J. Derbidge, *Kazhdan constants of cyclic groups*, Master's Thesis, California State University, Los Angeles, 2010.

[4] B. Elspas and J. Turner, *Graphs with circulant adjacency matrices*, J. Combin. Theory **9** (1970), 297-307.

[5] J. Gallian, *Contemporary Abstract Algebra, Seventh Edition*, Brooks/Cole, 2010

[6] S. Hoory, and N Linial, and A. Wigderson, *Expander graphs and their applications*, Bull. Amer. Math. Soc,**43** (2006), no. 4, 439—561.

[7] M. Krebs and A. Shaheen, *Expander Families and Cayley Graphs: A Beginner's Guide*, Oxford University Press, 2011.

[8] T.Y. Lam, *Representations of Finite Groups: A Hundred Years*, Notices of the AMS., **45** (1998), 361-372.

[9] P.P. Palfy, *Isomorphism problem for relational structures with a cyclic automorphism*, European J. Combin. **8** (1987), 35-43.

[10] J. Serre, *Linear representations of finite groups*, Graduate Texts in Mathematics, no. 42, Springer, 1977

APPENDIX A

C++ code for the Kazhdan constant

We give the C++ code that lets us calculate the Kazhdan constant for a cyclic group of even order with our particular generating set and conditions.

```
#include < iostream >
#include < cassert >
#include < vector >
#include < string >
#include < cmath >

using namespace std;
int main() {
int n=16;
cout<<"n="<< n <<"\n";
vector < int > Z;
vector < int > Max;
vector < int > data;
vector < int > Y;// this is vector Z in the other main one
vector < int > Top;// this is vector Max
vector < int > dee;//this is vector data

vector < int > H;
```

H.push_back(1);

H.push_back(2);

H.push_back(n/2-1);

// I should put vector D here for convince

$vector < int > D$; //call a int vector pos with 3 slots this is vector H

D.push_back(2);

D.push_back((n-4)/4);

D.push_back((n+4)/4);

// this spits out the vector

cout<< ”A={”;

for(unsigned int h=0; h< H.size(); h++)// H.size() gives the number of elements in the vector H.

{ if(h==0 || h==1) {

cout<<H[h]<< ”,”; } else { cout<<H[h]; } } cout<< ”}”;// and here is where it ends

cout<<”\n”;// We are couting the elements in H.

for(int j=1; j<=n/2; j++)

{ int x =((j) * H[0]) % n; int y =((j) * H[1]) % n; int z =((j) * H[2]) % n;

Z.push_back(x); Z.push_back(y); Z.push_back(z);

}// We will note that there are 3*(n/2) elements in Z[i] where i starts at 0.

```
cout<<”\n”;

for(unsigned i=0; i< Z.size();i++) { if(i%3 ==0) { int min; int value;

min = abs(n/2-Z[i]); /* assume x is the largest */

if (abs(n/2-Z[i+1]) < min) { /* if y is larger than max, assign y to max */
min = abs(n/2-Z[i+1]); } /* end if */

if (abs(n/2-Z[i+2]) < min) { /* if z is larger than max, assign z to max */ min
= abs(n/2-Z[i+2]); } /* end if */

int minz=min; if(n/2<Z[i]& abs(n/2-Z[i])==minz) { value=abs(n/2-minz)+2*minz;
}

else if(n/2<Z[i+1]& abs(n/2-Z[i+1])==minz) { value=abs(n/2-minz)+2*minz;
}

else if(n/2<Z[i+2]& abs(n/2-Z[i+2])==minz) { value=abs(n/2-minz)+2*minz;
}

else {

value=abs(n/2-minz);

}

Max.push_back(value);

cout<<”i=”<<i/3+1<<” ”<<”{”<<Z[i]<<”,”<<Z[i+1]<<”,”<<Z[i+2]<<”}”<<”
”<<”Max=”<<value<<”\n”; }

} cout<<”\n”;

cout<<”{”; for(unsigned i=0; i<Max.size(); i++) {

data.push_back(Max[i]);

if(i==0) { cout<<data[i]; } else {
```

cout<<”,”<<data[i]; }

}

cout<<”}”;

int j;

int change;

for(int i=0; i<=data.size()-2; i++)// this is where we sort the vector in an increasing fashion. { for(j=i+1; j<=data.size()-1; j++) { int w; do { change =0; for(i=0;i<data.size()-1; i++) { if(data[i]>data[i+1]) { w=data[i]; data[i]=data[i+1]; data[i+1]=w; change =1;

} } } while (change==1); } cout<<”\n”;

cout<<”This is your vectors sorted in increasing order”<<endl;

cout<<”{”; for(i=0;i<=data.size()-1;i++) { if(i==data.size()-1) { cout<<data[i]; } else cout<< data[i]<<”,”;

} //this is were the first sorting ends.

cout<<”}”<<”\n”;

if(data[0]<=n-data[data.size()-1]) { cout<<”The minimum element in this vector is ”<<data[0]; } else cout<<”The minimum element in this vector is ”<<n-data[data.size()-1];

//cout<<”The minimum element in this vector is ”<<data[0]<<”\n”;

cout<<”\n”;

}

cout<<”**”<<endl;

//We are again generating the elements in D. cout<< ”B={”; for(unsigned int h=0; h< D.size(); h++)// or we can use H.size { if(h==0|| h==1) {

cout<<D[h]<< ”,”; } else { cout<<D[h]; } } cout<<”}”;// and here is where it

cout<<”\n”;

cout<<”\n”;

for(int j=1; j<=n/2; j++)

{ int x =((j) * D[0]) % n; int y =((j) * D[1]) % n; int z =((j) * D[2]) % n;

Y.push_back(x); Y.push_back(y); Y.push_back(z);

}// We will note that there are 3*(n/2) elements in Z[i] where i starts at 0.

cout<<”\n”;

for(unsigned i=0; i< Y.size();i++) { if(i%3 ==0)//here is where it depends on the number of elements in H. { int ab;//min int walue;//value

ab = abs(n/2-Y[i]); /* assume x is the largest */

if (abs(n/2-Y[i+1]) < ab) { /* if y is larger than max, assign y to max */ ab = abs(n/2-Y[i+1]); } /* end if */

if (abs(n/2-Y[i+2]) <4 ab) { /* if z is larger than max, assign z to max */ ab = abs(n/2-Y[i+2]); } /* end if */

int mins=ab; if(n/2<Y[i]& abs(n/2-Y[i])==mins) { walue=abs(n/2-mins)+2*mins;

}

else if(n/2<Y[i+1]& abs(n/2-Y[i+1])==mins) { walue=abs(n/2-mins)+2*mins;

}

else if(n/2<Y[i+2]& abs(n/2-Y[i+2])==mins) { walue=abs(n/2-mins)+2*mins;

}

else {

walue=abs(n/2-mins);

}

Top.push_back(walue);

cout<<”i=”<<i/3+1<<” ”<<”{”<<Y[i]<<”,”<<Y[i+1]<<”,”<<Y[i+2]<<”}”<<” ”<<”Max=”<<walue<<”\n”; }

} cout<<”\n”;

cout<<”{”; for(unsigned i=0; i <Top.size(); i++) {

dee.push_back(Top[i]);

if(i==0) { cout<<dee[i]; } else {

cout<<”,”<<dee[i]; }

}

cout<<”}”;

for(int i=0; i¡=dee.size()-2; i++)// this is where we sort the vector in an increasing fashion. { for(j=i+1; j<=dee.size()-1; j++) { int w; do { change =0; for(i=0;i<dee.size()-1; i++) { if(dee[i]>dee[i+1]) { w=dee[i]; dee[i]=dee[i+1]; dee[i+1]=w; change =1;

} } } while (change==1); } cout<<”\n”;

cout¡¡ ”This is your vectors sorted in increasing order”¡¡endl;

cout<<”{”; for(i=0;i<=dee.size()-1;i++) { if(i==dee.size()-1) { cout<<dee[i]; } else cout<<dee[i]<<”,”;

} //this is were the first sorting ends.

cout<<”}”<<”\n”; //I am making the adjusments here if(dee[0]<=n-dee[dee.size()-

1]) { cout<<”The minimum element in this vector is ”<<dee[0]; } else cout<<”The minimum element in this vector is ”<<n-dee[dee.size()-1];

//cout<<”The minimum element in this vector is ”<<dee[0];

cout<<endl;

}

}